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Π-REGULAR VARIATION

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ABSTRACT. A function U: $R^+ \to R^+$ is said to be Π-regularly varying with exponent α if $U(x)x^{-\alpha}$ is nondecreasing and there exists a positive function L such that

$$\frac{U(\lambda x)/\lambda^\alpha - U(x)}{x^\alpha L(x)} \to \log \lambda \qquad (x \to \infty) \text{ for } \lambda > 0.$$

Suppose

$$\hat{U}(t) \coloneqq \int_0^\infty e^{-tx}\, dU(x) \text{ exists for } t > 0.$$

We prove that U is Π-regularly varying iff $\hat{U}$ is Π-regularly varying.

1. Introduction. First we give the definition of regular variation.

DEFINITION. A function U is said to be regularly varying with exponent ρ at infinity if it is real-valued, positive and measured on $(0, \infty)$ and if for each $\lambda > 0$

$$\lim_{x\to\infty} \frac{U(\lambda x)}{U(x)} = \lambda^\rho \quad \text{where } \rho \in R \text{ (notation } U(x) \in RV_\rho).$$

Regularly varying functions with exponent zero are called slowly varying. The theory of regularly varying functions has been developed by Karamata. For some basic facts see **[1]**, **[8]**, **[9]**.

A recent treatment of regular variation is also given in Seneta's book **[10]**. Karamata proved the following theorems on regular variation which are basic in this theory.

THEOREM A. *Suppose U: $R^+ \to R^+$ is Lebesgue summable on finite intervals.*

(i) *If U varies regularly at infinity with exponent $\beta > -1$ then*

$$\lim_{x\to\infty} \frac{xU(x)}{\int_0^x U(t)dt} = \beta + 1.$$

(ii) *If* $\lim_{x\to\infty}(xU(x)/\int_0^x U(t)dt) = \beta + 1$ *with* $\beta > -1$ *then* $U(x) \in RV_\beta$.
See, e.g., **[5**, *Theorem* 1.2.1].

The second theorem concerns the Laplace-Stieltjes transform: $\hat{U}(t) = \int_0^\infty e^{-ts}\, dU(s)$ of U. For a proof of this theorem the reader is referred to **[10**, Theorem 2.3].

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THEOREM B. *Suppose* U: $R^+ \to R^+$ *is nondecreasing, right-continuous* $U(0+) = 0$, $\hat{U}(t)$ *is finite for* $t > 0$. *For* $\beta \geqslant 0$ *the following assertions are equivalent*:

(i) $U(x) \in RV_\beta$;

(ii) $\hat{U}(1/x) \in RV_\beta$.

Both imply

(iii) $\lim_{x\to\infty}(U(x)/\hat{U}(1/x)) = 1/\Gamma(\beta + 1)$.

For nondecreasing functions U we can combine Theorems A and B using the notion of a fractional integral:

DEFINITION. ${}_\alpha U(x) = (1/\Gamma(\alpha + 1))\int_0^x (x - t)^\alpha \, dU(t)$ where $\alpha > 0$.

THEOREM C. *Suppose* U: $R^+ \to R^+$ *is nondecreasing and right-continuous*, $U(0+) = 0$ *and* $\hat{U}(t)$ *is finite for* $t > 0$. *For* $\alpha > 0$ *and* $\beta \geqslant 0$ *the following assertions are equivalent*:

(i) $U(x) \in RV_\beta$;

(ii) ${}_\alpha U(x) \in RV_{\alpha+\beta}$;

(iii) $\hat{U}(1/x) \in RV_\beta$.

They imply

(iv) ${}_\alpha U(x)/x^\alpha U(x) \to \Gamma(\beta + 1)/\Gamma(\alpha + \beta + 1)$ $(x \to \infty)$;

(v) $U(x)/\hat{U}(1/x) \to 1/\Gamma(\beta + 1)$ $(x \to \infty)$.

Remark that the case $\alpha = 1$ yields Theorem A(i) with $\beta \geqslant 0$. For arbitrary $\alpha > 0$ Theorem C can be proved by using Theorems A and B and the relation

$$ {}_\alpha\hat{U}(1/x) = x^\alpha \hat{U}(1/x), $$

since ${}_\alpha U(x)$ is nondecreasing.

In 1963 Bojanic and Karamata [2] studied the class of functions U for which

$$ \lim_{x\to\infty} \frac{U(\lambda x) - U(x)}{x^\sigma L(x)} $$

exists for some function $L(x)$ and showed that σ can be chosen such that $L(x)$ is slowly varying. In this paper we shall see that the Theorems A and B can be sharpened for functions U which satisfy the relation

$$ \lim_{x\to\infty} \frac{U(\lambda x)/\lambda^\sigma - U(x)}{x^\sigma L(x)} = \log \lambda $$

for some function $L(x)$ and $\sigma \geqslant 0$ fixed. For $\sigma = 0$ this relation defines the class Π.

THEOREM D. *Suppose* ϕ: $R^+ \to R$ *is nondecreasing. Then the following three statements are equivalent*:

(i) *There exist functions* a: $R^+ \to R^+$ *and* b: $R^+ \to R$ *such that for all positive* x

$$ \lim_{t\to\infty} \frac{\phi(tx) - b(t)}{a(t)} = \log x; $$

(ii) *there exists a slowly varying function* L *such that*

$$ \phi(x) = L(x) + \int_1^x L(t)/t \, dt; $$

(iii) *there exists a slowly varying function* L_0 *such that*

$$\phi(x) = L_0(x) + \int_0^x L_0(t)/t\, dt.$$

Moreover if a function ϕ *satisfies the conditions of this theorem then*

$$a(x) \sim L(x) \sim \phi(xe) - \phi(x) \sim \frac{1}{x}\int_0^x s\, d\phi(s) \sim L_0(x) \qquad (x \to \infty)$$

(*see* [5, *Theorem* 1.4.1]).

We call the function $a(x)$ the auxiliary function of $\phi(x)$. This function is (of course) determined up to asymptotic equivalence.

DEFINITION. A function ϕ which satisfies the conditions of Theorem D is said to belong to the class Π. It can be shown that the class Π is a proper subclass of the slowly varying functions (see [5, Corollary 1.4.1]). From Theorem D we can see that if $\phi(x) \in \Pi$ with auxiliary functions $a(x)$ and $[\phi(x) - \phi_1(x)]/a(x) \to c$ $(x \to \infty)$ where $c \in R$ is a constant and $\phi_1(x)$ a nondecreasing function, then $\phi_1(x) \in \Pi$ with auxiliary function $a(x)$.

In this paper we generalize the following theorem (see [6]).

THEOREM E. *Suppose* ϕ: $R^+ \to R^+$ *is nondecreasing*, $\phi(0+) = 0$ *and* $\hat{\phi}(s)$ *is finite for* $s > 0$. *Then the following statements are equivalent*:

(i) $\phi(x) \in \Pi$;

(ii) $\hat{\phi}(1/x) \in \Pi$;

Both imply

(iii) $(\phi(x) - \hat{\phi}(1/x))/(1/x)\int_0^x s\, d\phi(s) \to \gamma$ $(x \to \infty)$.

We give a second order version of Karamata's Theorems A and B for nondecreasing functions U. A necessary and sufficient condition for a function to obey the second order relation is formulated in the following definition.

DEFINITION. $U \in \Pi RV_\alpha$ iff $U(x)/x^\alpha \in \Pi$ where $\alpha \in R$.

If $U \in \Pi RV_\alpha$ then we say that L is the auxiliary function of U if L is the auxiliary function of $U(x)/x^\alpha \in \Pi$. We call the function U Π-regularly varying with exponent α. The Π-varying functions with exponent α form a subclass of RV_α.

2. Results. Our result is the following theorem.

THEOREM 1. *Suppose* $\alpha > 0$, $\beta \geqslant 0$, U: $R^+ \to R^+$, $U(x)/x^\beta$ *nondecreasing*, $\lim_{x\downarrow 0}(U(x)/x^\beta) = 0$, *and* $\hat{U}(t)$ *exists for* $t > 0$. *Then the following statements are equivalent*:

(i) $U(x) \in \Pi RV_\beta$;

(ii) ${}_\alpha U(x) \in \Pi RV_{\alpha+\beta}$;

(iii) $\hat{U}(1/x) \in \Pi RV_\beta$.

They imply

$$\text{(iv)} \qquad \frac{(\Gamma(\beta+1)/\Gamma(\alpha+\beta+1))U(x) - {}_\alpha U(x)/x^\alpha}{x^{\beta-1}\int_0^x s\, d(U(s)/s^\beta)} \to -\frac{\partial}{\partial\beta}\left(\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\right)$$

$(x \to \infty)$,

$$\text{(v)}\qquad \frac{U(x) - (1/\Gamma(\beta+1))\hat{U}(1/x)}{x^{\beta-1}\int_0^x s\, d(U(s)/s^\beta)} \to -\psi(\beta+1) \qquad (x\to\infty)$$

where $\psi(x) = (d/dx)\log\Gamma(x)$.

Conversely if (iv) *with* $\alpha \in (0, 1]$, $\beta > 0$ *then* (i) *and if* (v) *with* $\beta > 1$ *then* (i).

PROOF. (i) $\to$ (iv) and (i) $\to$ (ii). We write

$$U(x) = x^\beta\left(L(x) + \int_0^x \frac{L(t)}{t}\, dt\right)$$

with

$$L(x) = \frac{1}{x}\int_0^x s\, d\frac{U(s)}{s^\beta} \in RV_0^{(\infty)}.$$

Then

$$\frac{({}_\alpha U(x)/x^\alpha) - (\Gamma(\beta+1)/\Gamma(\alpha+\beta+1))U(x)}{x^\beta L(x)}$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1} t^\beta \frac{(U(tx)/t^\beta x^\beta) - (U(x)/x^\beta)}{L(x)}\, dt$$

$$\to \frac{1}{\Gamma(\alpha)}\int_0^1 \log t (1-t)^{\alpha-1} t^\beta\, dt \qquad (x\to\infty).$$

The last step is justified since by substituting the expression for $U(x)$ we find

$$\frac{1}{\Gamma(\alpha)}\left[\int_0^1 (1-t)^{\alpha-1} t^\beta \left\{\frac{L(tx)}{L(x)} - 1\right\} dt - \int_0^1 (1-t)^{\alpha-1} t^\beta \int_t^1 \frac{L(sx)}{L(x)}\frac{ds}{s}\, dt\right]$$

and

$$\frac{s^\varepsilon x^\varepsilon L(sx)}{x^\varepsilon L(x)} \to s^\varepsilon \qquad (x\to\infty)$$

uniformly on (0, 1) where $\varepsilon > 0$ (see de Haan [5]). Now (iv) and (i) imply (ii) as mentioned in the introduction.

(i) $\to$ (v) and (i) $\to$ (iii). We write $U(x) = x^\beta L(x) + K(x)$ where $K(x) = x^\beta\int_0^x (L(t)/t)dt$. By Karamata's Theorem B we have

$$x^\beta L(x) - \frac{1}{\Gamma(\beta+1)}\int_0^\infty e^{-t/x}\, d(t^\beta L(t)) = o(x^\beta L(x)) \qquad (x\to\infty).$$

Substituting the expression for $K(x)$ we find

$$\frac{K(x) - \hat{K}(1/x)/\Gamma(\beta+1)}{x^\beta L(x)} = \int_0^1 \frac{L(tx)}{L(x)}\frac{dt}{t}$$

$$- \frac{1}{\Gamma(\beta+1)}\int_0^\infty e^{-t} t^\beta \int_0^t \frac{L(ux)}{L(x)}\frac{du}{u}\, dt$$

$$= -\frac{1}{\Gamma(\beta+1)}\int_0^1 e^{-t} t^\beta \int_1^t \frac{u^\varepsilon x^\varepsilon L(ux)}{x^\varepsilon L(x)}\frac{du}{u^{1+\varepsilon}}\, dt$$

$$- \frac{1}{\Gamma(\beta+1)}\int_1^\infty e^{-t} t^\beta \int_1^t \frac{u^{-\varepsilon} x^{-\varepsilon} L(ux)}{x^{-\varepsilon} L(x)}\frac{du}{u^{1-\varepsilon}}\, dt = (*)$$

since $\Gamma(\beta + 1) = \int_0^\infty e^{-t} t^\beta \, dt$. Since

$$\frac{u^\varepsilon x^\varepsilon L(ux)}{x^\varepsilon L(x)} \to u^\varepsilon \; (x \to \infty) \quad \text{uniformly on } (0, 1)$$

and

$$\frac{u^{-\varepsilon} x^{-\varepsilon} L(ux)}{x^{-\varepsilon} L(x)} \to u^{-\varepsilon} \; (x \to \infty) \quad \text{uniformly on } (1, \infty)$$

(see [5, Corollary 1.2.1.4]) we find

$$(*) \to -\frac{1}{\Gamma(\beta + 1)} \int_0^\infty e^{-t} t^\beta \log t \, dt = -\psi(\beta + 1) \qquad (x \to \infty).$$

This proves (i) $\to$ (v). Now we have analogously that (v) and (i) imply (iii).

(ii) $\to$ (iii) follows immediately since ${}_\alpha\hat{U}(1/x) = x^\alpha \hat{U}(1/x)$ and we can use (i) $\to$ (iii).

(i) $\leftrightarrow$ (iii). Writing $V(x) = U(x)/x^\beta$ we have by Proposition P4 in [7]

$$V(x) \in \Pi \text{ iff } \int_0^x t^\beta \, dV(t) \in RV_\beta \quad \text{where } \beta > 0.$$

Or

$$U(x) \in \Pi RV_\beta \text{ iff } U(x) - \int_0^x \frac{U(t)}{t} \, dt \in RV_\beta.$$

This is equivalent to

$$\hat{U}(1/x) - \beta x \hat{K}(1/x) \in RV_\beta \quad \text{where } K(x) = U(x)/x.$$

The last statement is equivalent to $\hat{U}(1/x) \in \Pi RV_\beta$, since $x\hat{K}(1/x) = \int_0^x (\hat{U}(1/t)/t) dt$. (Both sides have the same derivative.) The case $\beta = 0$ is the result of Theorem E.

(iv) $\to$ (i). As in the proof of (i) $\to$ (iv) we write

$$U(x) = x^\beta \left\{ L(x) + \int_0^x \frac{L(t)}{t} \, dt \right\}.$$

Substituting this expression in (iv) and rearranging we see that (iv) is equivalent to

$$\frac{1}{\Gamma(\alpha)} \int_0^x \left(1 - \frac{t}{x}\right)^{\alpha - 1} \left(\frac{t}{x}\right)^{\beta + 1} L(t) \frac{dt}{t} + \frac{1}{\Gamma(\alpha)} \int_0^x \int_{t/x}^1 (1 - u)^{\alpha - 1} u^\beta \, du L(t) \frac{dt}{t}$$
$$- \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} \int_0^x L(t) \frac{dt}{t} \sim \xi L(x) \qquad (x \to \infty),$$

where $\xi = (\Gamma(\beta + 1)/\Gamma(\alpha + \beta + 1)) + (d/d\beta)\{\Gamma(\beta + 1)/\Gamma(\alpha + \beta + 1)\}$. Or $\int_0^\infty L(t) k(x/t) \, dt/t \sim \xi L(x)$ $(x \to \infty)$ where the kernel k is defined by

$$k(1/x) = \frac{x}{\Gamma(\alpha)} \left\{ (1 - x)^{\alpha - 1} x^\beta - \frac{1}{x} \int_0^x (1 - u)^{\alpha - 1} u^\beta \, du \right\}$$

for $x < 1$ and 0 for $x \geq 1$. For $\alpha \in (0, 1]$ and $\beta > 0$ the kernel is nonnegative since $(1 - x)^{\alpha - 1} x^\beta$ is increasing on $(0, 1)$. Moreover we have $\lim_{x \to \infty; t \to 1+} \inf(L(tx)/L(x)) \geq 1$ since $xL(x)$ is nondecreasing. Application of Theorem 6.2 in [3] then gives the result since $\hat{k}(\rho) = \int_0^1 k(1/t) t^{\rho - 1} \, dt$ is decreasing for $\rho > -\beta - 1$ and so $\hat{k}(\rho) = \xi$ only if $\rho = 0$.

(v) $\to$ (i). We define $L(x)$ as in the proof of (iv) $\to$ (i). Here we can reformulate (v) as follows:

$$\int_0^\infty k\left(\frac{x}{t}\right)L(t)\frac{dt}{t} \sim \xi L(x) \qquad (x \to \infty),$$

where $\xi = 1 + \psi(\beta + 1)$ and the kernel k is given by

$$k\left(\frac{1}{x}\right) = \frac{1}{\Gamma(\beta + 1)}x^\beta e^{-x} - \frac{1}{x}\int_0^x u^\beta e^{-u}\,du + 1 - I_{(0,1)}(x).$$

If $\beta > 1$ this kernel is positive for all $x > 0$, since the term $x^\beta e^{-x}$ is increasing on $(0, \beta)$. Here we can also apply Theorem 6.2 in [**3**].

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