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EXTENSIONS OF PURE POSITIVE FUNCTIONALS ON BANACH *-ALGEBRAS

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ABSTRACT. A known extension theorem for pure states on a Banach *-algebra with isometric involution is shown to hold for the wider class of Banach *-algebras with arbitrary, possibly discontinuous, involutions.

Let A be a Banach *-algebra with isometric involution and bounded approximate identity $\{e_\alpha\}$, and B a closed *-subalgebra of A containing $\{e_\alpha\}$. In [3] G. Maltese proved that if f is a pure state on B, then f admits a pure state extension to A if and only if f admits a positive linear extension to A. Our purpose here is to extend this result to Banach *-algebras with arbitrary, possibly discontinuous, involutions.

For basic definitions and results from the theory of Banach *-algebras and their representations see [1], [2], or [4].

The following lemma handles the case when the algebra A contains an identity.

LEMMA 1. *Let A be a unital Banach *-algebra, B a closed *-subalgebra of A containing the identity e, and suppose that f is a pure positive linear functional on B. Then f can be extended to a pure positive linear functional on A if and only if f has a positive linear extension to A.*

PROOF. We may assume without loss of generality that $f(e) = 1$. Indeed, if $\lambda > 0$, then λf is pure and positive if f is pure and positive. Our proof will be given in two steps:

I. A has continuous involution;

II. A has arbitrary involution.

PROOF OF I. Let P_A denote the set of positive functionals g on A satisfying $g(e) = 1$. Define P_B similarly. It is well known that a functional in P_A (or P_B) is pure (pure on B) if and only if it is an extreme point of P_A (P_B). Suppose, now, that f has a positive linear extension to A, and set $X = \{g \in P_A: g|_B = f\}$; i.e., X is the set of all positive extensions of f. Then X is nonempty by assumption, and it is clearly convex. We show that X is compact in the relative weak *-topology. By the Banach-Alaoglu theorem it suffices to show that X is weak *-closed and norm bounded. Suppose that $\{g_\alpha\}$ is a net in X and that $g_\alpha \to g$. Then by the definition of the weak *-topology, $g_\alpha(x) \to g(x)$ for every $x \in A$; thus, if $x \in B$, then

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$g(x) = \lim_\alpha g_\alpha(x) = \lim_\alpha f(x) = f(x)$ which implies $g \in X$. Therefore, X is weak *-closed. Now let $g \in X$ be arbitrary; since $x \to x^*$ is continuous there exists $k > 0$ such that $\|x^*\| \leq k\|x\|$ for all x in A. Then, by [4, pp. 214, 219],

$$|g(x)|^2 \leq g(e)g(x^*x) \leqslant g(e)g(e)\nu(x^*x) \leq g(e)^2\|x^*x\| \leq g(e)^2k\|x\|^2 = k\|x\|^2,$$

where $\nu(\cdot)$ denotes the spectral radius. Hence $\|g\| \leq \sqrt{k}$ and X is norm bounded. The Krein-Milman theorem now implies that X has extreme points. We denote the set of extreme points of X, P_A, and P_B by $E(X)$, $E(P_A)$, and $E(P_B)$ respectively. Verification of the equality $E(X) = X \cap E(P_A)$ will complete the proof of Part I. Our proof follows that given in [3, p. 503].

It is clear that $X \cap E(P_A) \subseteq E(X)$. Let $g \in E(X)$ and suppose $g = \frac{1}{2}(\phi + \psi)$, where $\phi, \psi \in P_A$. Then, taking restrictions, we obtain $f = \frac{1}{2}(\phi|_B + \psi|_B)$. But $\phi|_B$ and $\psi|_B$ are in P_B, and since $f \in E(P_B)$, it follows that $f = \phi|_B = \psi|_B$ which implies that ϕ and ψ are in X. Since g is an extreme point of X we have $g = \phi = \psi$ which means that $g \in E(P_A)$. Hence $E(X) \subseteq X \cap E(P_A)$.

Proof of II. We now allow the involution to be arbitrary. If J denotes the Jacobson radical of A, then A/J is a semisimple Banach *-algebra which, by Johnson's uniqueness of the norm theorem [1, p. 130], has continuous involution. Hence the closure $((B + J)/J)^-$ in A/J of the *-subalgebra $(B + J)/J$ is a Banach *-subalgebra of A/J containing the identity $e + J$.

Let f' be the positive extension of f to A, and define a function $\bar{f}'$: $A/J \to \mathbf{C}$ by $\bar{f}'(x + J) = f'(x)$. We note that $\bar{f}'$ is well defined since f' is representable [4, p. 216] and J is contained in the reducing ideal. Furthermore, $\bar{f}'$ is linear and positive and is therefore continuous since A/J has an identity. Moreover, if $b \in B$, then $\bar{f}'(b + J) = f'(b) = f(b)$. Let

$$\bar{f} = \bar{f}'|_{((B+J)/J)^-}.$$

Then $\bar{f}$ is a continuous positive linear functional on $((B + J)/J)^-$ and $\bar{f}(b + J) = f(b)$ for every $b \in B$. We assert that $\bar{f}$ is pure. Indeed, let $\bar{g}$ be an arbitrary positive functional on $((B + J)/J)^-$ satisfying $\bar{g} \leq \bar{f}$. Then $\bar{g}(b^*b + J) \leq \bar{f}(b^*b + J) = f(b^*b)$ for every $b \in B$. Define a positive functional g on B by $g(b) = \bar{g}(b + J)$. Clearly $g \leq f$, and since f is pure, it follows that $g = \lambda f$, where $0 < \lambda \leq 1$. Hence, $\bar{g} = \lambda\bar{f}$ on $(B + J)/J$. But $\bar{g}$ and $\bar{f}$ are both continuous, and thus it follows that $\bar{g} = \lambda\bar{f}$ on all of $((B + J)/J)^-$; therefore, $\bar{f}$ is pure.

By part I, $\bar{f}$ has a pure positive extension to A/J which we denote by h. Define $h' : A \to \mathbf{C}$ by $h'(x) = h(x + J)$. Then h' is a positive functional on A and if $b \in B$, then $h'(b) = h(b + J) = \bar{f}(b + J) = f(b)$. It remains only to show that h' is pure. Let g' be a positive functional on A satisfying $g' \leq h'$. Then $g'(x^*x) \leq h'(x^*x) = h(x^*x + J)$. Define a functional g on A/J by $g(x + J) = g'(x)$; g is well defined since g' is representable. Clearly g is positive and $g \leq h$; but h is pure, so $g = \lambda h$ which implies $g' = \lambda h'$. Hence h' is pure and the proof is complete.

The next lemma is well known from Banach *-algebras with isometric involution (see [2, 2.2.10, p. 34]). We give a simple proof for the case of an arbitrary involution. In what follows we assume that all bounded approximate identities are bounded by one.

LEMMA 2. *Let A be a Banach *-algebra with bounded approximate identity $\{e_\alpha\}$, π a nondegenerate *-representation of A on a Hilbert space H, and let I denote the identity operator on H. Then $\lim_\alpha \pi(e_\alpha) = I$, where the limit is in the strong operator topology.*

PROOF. For each $x \in A$ we have $\|\pi(e_\alpha) - \pi(x)\| \leqslant \|\pi\| \cdot \|e_\alpha x - x\| \underset{\alpha}{\to} 0$. Hence $\|\pi(e_\alpha)\pi(x)\xi - \pi(x)\xi\| \underset{\alpha}{\to} 0$ for every $x \in A$ and every $\xi \in H$. Since π is nondegenerate, the set $\pi(A)H$ is dense in H. Now let $\eta \in H$ be arbitrary, $\varepsilon > 0$, and set $M = \max\{\|\pi\|, 1\}$. Then there exists $\xi \in H$ and $x \in A$ such that $\|\pi(x)\xi - \eta\| < \varepsilon/3M$ and there exists α_0 such that $\alpha \geqslant \alpha_0$ implies

$$\|\pi(e_\alpha)\pi(x)\xi - \pi(x)\xi\| < \varepsilon/3.$$

Then

$$\begin{aligned}\|\pi(e_\alpha)\eta - \eta\| &\leqslant \|\pi(e_\alpha)\eta - \pi(e_\alpha)\pi(x)\xi\| \\ &\quad + \|\pi(e_\alpha)\pi(x)\xi - \pi(x)\xi\| + \|\pi(x)\xi - \eta\| \\ &< \|\pi\| \cdot \|e_\alpha\| \cdot \|\eta - \pi(x)\xi\| + \varepsilon/3 + \varepsilon/3M \leqslant \varepsilon\end{aligned}$$

completing the proof.

THEOREM 3. *Let A be a Banach *-algebra with bounded approximate identity $\{e_\alpha\}$ and suppose B is a closed *-subalgebra of A containing $\{e_\alpha\}$. Let f be a pure positive linear functional on B admitting a positive linear extension f' to A. Then f has a pure positive linear extension to A.*

PROOF. Since f and f' are representable, we can write $f(b) = (\pi(b)\xi|\xi)$ and $f'(x) = (\pi'(x)\xi'|\xi')$ for all $b \in B$, $x \in A$, and suitable vectors ξ and ξ' in the respective spaces of π and π'. Then, by Lemma 2, $(\xi|\xi) = \|\xi\|^2 = \lim_\alpha f(e_\alpha) = \lim_\alpha f'(e_\alpha) = \|\xi'\|^2 = (\xi'|\xi')$. Let A_e and B_e denote the Banach *-algebras obtained from A and B respectively by adjoining identities. Define *-representations $\bar{\pi}'$ and $\bar{\pi}$ of A_e and B_e respectively by $\bar{\pi}'[(x, \lambda)] = \pi'(x) + \lambda I$ and $\bar{\pi}[(b, \lambda)] = \pi(b) + \lambda I$, where I denotes the identity operator. Let $\bar{f}'[(x, \lambda)] = (\bar{\pi}'[(x, \lambda)]\xi'|\xi')$ and $\bar{f}[(b, \lambda)] = (\bar{\pi}[(b, \lambda)]\xi|\xi)$. Then $\bar{f}'$ and $\bar{f}$ are positive functionals on A_e and B_e respectively and

$$\begin{aligned}\bar{f}'[(b, \lambda)] &= (\bar{\pi}'[(b, \lambda)]\xi'|\xi') = (\pi'(b)\xi'|\xi') + \lambda(\xi'|\xi') \\ &= (\pi(b)\xi|\xi) + \lambda(\xi|\xi) = (\bar{\pi}[(b, \lambda)]\xi|\xi) = \bar{f}[(b, \lambda)]\end{aligned}$$

for every $(b, \lambda) \in B_e$. Now f pure implies that π is irreducible [2, 2.5.4, p. 43]; hence $\bar{\pi}$ is irreducible and thus $\bar{f}$ is pure. By Lemma 1, $\bar{f}$ has a pure positive extension, say g, to A_e. Hence there exists an irreducible *-representation π_g and a cyclic vector ξ_g such that $g[(x, \lambda)] = (\pi_g[(x, \lambda)]\xi_g|\xi_g)$. So $\pi_g|_A$ is also irreducible, and therefore the functional g_A defined on A by $g_A(x) = (\pi_g|_A(x)\xi_g|\xi_g)$ is a pure positive functional on A. Moreover, $g_A(b) = g[(b, 0)] = \bar{f}[(b, 0)] = (\bar{\pi}[(b, 0)]\xi|\xi) = (\pi(b)\xi|\xi) = f(b)$.

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