

ELEMENTARY PROOF OF THE RUDIN-CARLESON
AND THE F. AND M. RIESZ THEOREMS

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ABSTRACT. A very elementary proof is given of the theorem that on a set of measure zero on T , any continuous function is equal to a continuous function of analytic type. The same elementary method proves that a measure of analytic type is absolutely continuous.

A complex Borel measure μ on T , in particular an $f \in L^1(T)$, is said to be of analytic type if

$$a_n = (2\pi)^{-1} \int_T e^{-int} d\mu(t) = 0, \quad n = -1, -2, \dots$$

The theorems mentioned in the title are:

RUDIN-CARLESON THEOREM. *Let F be a closed subset of T of Lebesgue measure zero. If φ is a continuous function on F , then there is a continuous function f , of analytic type, such that*

$$\begin{aligned} f(t) &= \varphi(t), & t \in F, \\ \sup_{t \in T} |f(t)| &\leq M \sup_{t \in F} |\varphi(t)| \end{aligned} \quad (*)$$

where M is a constant. (Rudin proves that $M = 1$. See [8] and [1].)

THE FIRST F. AND M. RIESZ THEOREM. *If the function f in $L^1(T)$ is of analytic type and if f vanishes on a set S^* of positive measure, then $f = 0$.*

THE SECOND F. AND M. RIESZ THEOREM. *If a complex Borel measure μ on T is of analytic type, then μ is absolutely continuous (with respect to Lebesgue measure). See [7].*

The proofs of these theorems most often use boundary values of functions analytic in the unit disc and the theory of H^p -spaces. For the Second F. and M. Riesz Theorem, for example, see three variants in [3], [5] and [9]; other proofs of that theorem use Hilbert-space theory: see e.g. [2] and [4]; a direct short proof is given in [6].

The aim of the present paper is to present a method which gives an elementary proof of all the above theorems.

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LEMMA. Let F be a closed subset of T of measure zero and φ a continuous function on F . Given $\varepsilon > 0$ and an open set $G \supset F$ there is a continuous function g of analytic type such that

$$\begin{aligned} \sup_{t \in F} |g(t) - \varphi(t)| &< \varepsilon \sup_{t \in F} |\varphi(t)|, \\ |g(t)| &< \varepsilon, \quad t \notin G, \\ \sup_{t \in T} |g(t)| &< 3 \sup_{t \in F} |\varphi(t)|. \end{aligned} \quad (**)$$

PROOF. Without loss of generality we may assume that $\sup_{t \in F} |\varphi(t)| = 1$ and also that φ is a trigonometric polynomial

$$\varphi(t) = \sum_{|k| < m} \alpha_k e^{ikt}$$

such that

$$|\varphi(t)| < \varepsilon/3, \quad t \notin G.$$

Let $e^{-A} = \varepsilon$ and let h be a continuous function on T , lying between $-2A$ and 2ε , such that

$$|h(t) + 2A| < \varepsilon, \quad t \in F.$$

Since $m(F) = 0$ we may take $\|h\|_1$ arbitrarily small and hence we may suppose $\hat{h}(k) = 0$, $|k| < m$. Take a Fejér sum p of h such that $|p(t) + 2A| < \varepsilon$, $t \in F$. We write

$$p(t) = \sum_{k < -m} \beta_k e^{ikt} + \sum_{k > m} \beta_k e^{ikt} = p^-(t) + p^+(t)$$

where

$$p^+(t) = \sum_{k > m} \beta_k e^{ikt}.$$

We have $\operatorname{Re}(p^+) = p/2 < \varepsilon$. Put now

$$g(t) = \varphi(t)[1 - e^{p^+(t)}].$$

The expansion of $[1 - e^{p^+(t)}]$ is of the form $\sum_{k > m} \gamma_k e^{ikt}$. The function g is therefore continuous of analytic type. We have

$$|g(t) - \varphi(t)| = |\varphi(t)| |e^{p^+(t)}| < e^{p/2} < e^{-A+\varepsilon} < 2\varepsilon \quad (t \in F).$$

Moreover

$$|g(t)| < |\varphi(t)| |1 - e^{p^+(t)}| < 1 + e^\varepsilon < 3 \quad (t \in T).$$

$$|g(t)| < (\varepsilon/3)3 = \varepsilon \quad (t \notin G).$$

The Lemma is now proved.

PROOF OF THE RUDIN-CARLESON THEOREM. $\varepsilon < \frac{1}{4}$ being fixed, denote by $\gamma(\varphi)$ any continuous function of analytic type associated to φ by the Lemma. Starting with $\varphi_0 = \varphi$ we put $\varphi_{m+1} = \varphi_m - \gamma(\varphi_m)$. We have

$$\sup_F |\varphi_{m+1}| < \varepsilon \sup_F |\varphi_m| < \dots < \varepsilon^{m+1} \sup_F |\varphi_0|,$$

$$\sup_T |\gamma(\varphi_m)| < 3 \sup_F |\varphi_m| < 3\varepsilon^m \sup_F |\varphi_0|.$$

The series $\sum_{m=0}^{\infty} \gamma(\varphi_m)$ is therefore uniformly convergent on T ; its sum f is of analytic type and satisfies the relation $f(t) = \varphi(t)$ ($t \in F$). Moreover

$$\sup_T |f(t)| < 3(1 - \epsilon)^{-1} \sup |\varphi_0| < 4 \sup_F |\varphi|.$$

The theorem is now proved.

REMARK. The factor M in the estimate (*) can easily be reduced to $1 + \epsilon$. In fact, given an open set $G \supset F$ and using (**) we can manage to have

$$|f(t)| < \epsilon \quad (t \notin G).$$

By the continuity of f , there is an open set $G' \supset F$ such that $G' \subset G$ and $|f(t)| < 1 + \epsilon$ ($t \in G'$). Thus we can have

$$|f(t)| \geq 1 + \epsilon \quad \text{only if } t \in G \setminus G'.$$

Starting with G' we get f' coinciding with φ on F , bounded by 4 where $|f'(t)| > 1 + \epsilon$ only if $t \in G' \setminus G''$ for an appropriate $G'' \supset F$, with $G'' \subset G'$. Observing that the sets $G \setminus G', G' \setminus G'', G'' \setminus G''', \dots$ are disjoint and taking an arithmetic mean we get a function bounded everywhere by $1 + 2\epsilon$.

PROOF OF THE FIRST F. AND M. RIESZ THEOREM. It is sufficient to prove that

$$a_0 = (2\pi)^{-1} \int_T f(t) dt = 0$$

for, applying the same process to the function $e^{-it}f(t)$, we deduce $a_1 = 0$, and next $a_2 = 0, \dots$ and finally $f = 0$. We shall follow the same pattern of proof as for the Rudin-Carleson Theorem.

Denote by S the set $\{t \in T: f(t) \neq 0\}$. Given $\epsilon > 0$ let $e^{-A} = \epsilon$ and let h be a bounded real function equal to $-2A$ on S and such that $\hat{h}(0) = 0$. There are such functions since $m(S^*) > 0$. Let p_n be the sequence of Fejér polynomials of h . We write as before

$$p_n(t) = \sum_{k < 0} \beta_k e^{ikt} + \sum_{k > 0} \beta_k e^{ikt} = p_n^-(t) + p_n^+(t)$$

where

$$p_n^+(t) = \sum_{k > 0} \beta_k e^{ikt}.$$

Then, boundedly,

$$\text{Re}(p_n^+(t)) = \frac{1}{2} p_n(t) \rightarrow \frac{1}{2} h(t) = -A \quad \text{a.e. on } S.$$

Put now

$$g_n(t) = f(t)[1 - e^{p_n^+(t)}].$$

The expansion of g_n is of the form $\sum_{k > 0} \gamma_k e^{ikt}$ and therefore $\int g_n dt = 0$. Hence

$$\begin{aligned} |2\pi a_0| &= \left| \int f \right| = \left| \int (f - g_n) \right| < \left| \int f e^{p_n^+} \right| \\ &< \int_S |f| e^{p_n/2} \rightarrow e^{-A} \int |f| = \epsilon \|f\|_1. \end{aligned}$$

Since ϵ is arbitrary we have $a_0 = 0$ and the theorem is proved.

PROOF OF THE SECOND F. AND M. RIESZ THEOREM. We may assume $a_0 = 0$. Let F be a closed set of measure zero. Choose a decreasing sequence of open sets $G_n \supset F$ such that $\bigcap G_n = F$, and by the Lemma a sequence of functions g_n of analytic type, such that

$$\begin{aligned} |1 - g_n(t)| &< 1/n, & t \in F, \\ |g_n| &< 3; & |g_n(t)| < 1/n \text{ for } t \notin G_n. \end{aligned}$$

Then, boundedly, $g_n \rightarrow \chi_F$ (characteristic function of F). Hence $0 = \int g_n d\mu \rightarrow \int \chi_F d\mu = \mu(F)$. This proves that μ is absolutely continuous.

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