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CONTINUITY OF THE SPECTRUM AND SPECTRAL RADIUS

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ABSTRACT. Let A be a Banach algebra containing an element x. Topological conditions on the spectrum of x are given which are necessary and sufficient to ensure the continuity of the spectrum or spectral radius at x.

1. Introduction. In this paper **C** denotes the field of complex numbers, and **K** the metric space of nonempty compact subsets of **C** endowed with the Hausdorff metric Δ. If $K_1, K_2 \in \mathbf{K}$, then

$$\Delta(K_1, K_2) = \max\Big(\sup_{\lambda \in K_2} d(\lambda, K_1), \sup_{\lambda \in K_1} d(\lambda, K_2) \Big)$$

where $d(\lambda, K_1) = \inf_{\mu \in K_1} |\lambda - \mu|$.

A will denote a Banach algebra, which we will always assume is unital without loss of generality. For $x \in A$, $\sigma(x)$ and $r(x)$ denote the spectrum of x and spectral radius of x respectively. We are interested in determining the points in A at which the spectrum $\sigma: A \to \mathbf{K}$, $x \to \sigma(x)$ or the spectral radius $r: A \to [0, \infty)$, $x \to r(x)$ are continuous. Newburgh **[5]** initiated the study of this problem in 1951. A famous example, due to Kakutani **[4**, pp. 248–249], shows that the spectral radius is discontinuous at certain elements in the C^*-algebra $B(H)$ of all bounded linear operators on a separable Hilbert space H. Recently Conway and Morrel **[3]** have given necessary and sufficient conditions for the continuity of σ and r at T in $B(H)$. Their results depend on a deep theorem of Apostol and Morrel **[1**, Theorem 3.1] a special case of which we now state, as it will be used in the sequel.

THEOREM 1. *If T is a normal operator on H, and S is a closed subset of* **C** *which meets all the components of $\sigma(T)$, then there is a sequence T_n in $B(H)$ converging to T in norm for which $\sigma(T_n) \subseteq S$ $(n > 0)$.*

Let A be a Banach algebra, $x \in A$, and U an open subset of **C**. There are a number of results, due essentially to Newburgh **[5]** which we shall need later.

THEOREM 2. *If $U \supseteq \sigma(x)$, then there exists $\delta > 0$ such that $\|y - x\| < \delta$ implies that $U \supseteq \sigma(y)$.*

(This property of σ is referred to as *upper semicontinuity*.)

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THEOREM 3. *If U contains a component of $\sigma(x)$, then there exists $\delta > 0$ such that $\|y - x\| < \delta$ implies that U contains a component of $\sigma(y)$.*

It is trivial to see that r is always upper semicontinuous, and that, since $r(x) = \Delta(\sigma(x), \{0\})$, if σ is continuous at x, so is r. An interesting result due to Aupetit [2] states that r is uniformly continuous on A if and only if $A/\mathrm{rad}(A)$ is commutative ($\mathrm{rad}(A)$ denotes the Jacobson radical of A).

2. Continuity of r and σ. Throughout this section A is a (unital) Banach algebra, $x \in A$, and T a normal operator on a separable infinite dimensional Hilbert space H. Let $K \in \mathbf{K}$, and put $\alpha(K) = \sup\{\inf_{\lambda\in\omega}|\lambda|: \omega$ is a component of $K\}$, and $r(K) = \sup_{\lambda\in K}|\lambda|$. So $r(K) \geqslant \alpha(K)$. We assume $\sigma(T) = K$.

DEFINITION 1. The set K is an *r-set* (resp. *σ-set*) if for every Banach algebra A, and x in A with $\sigma(x) = K$, the spectral radius r (resp. the spectrum σ) is continuous at x in A. A well-known result of Newburgh [5] can be restated by saying that if K is totally disconnected it is a σ-set.

(Note that this is true for the spectrum $\sigma(S)$ of a compact or Riesz operator S on a Banach space X, so that σ is continuous at S in the Banach algebra $B(X)$ of bounded linear operators on X.)

Assume K is a member of $\mathbf{K}$.

PROPOSITION 1. *The following are equivalent statements*:
(i) *K is an r-set*;
(ii) $\alpha(K) = r(K)$;
(iii) *T is a point of continuity of r in $B(H)$.*

PROOF. (ii) $\Rightarrow$ (i). If $\alpha(K) = r(K)$ and $K = \sigma(x)$ for x in some Banach algebra A then, for any $\varepsilon > 0$, let $U = \{\lambda \in \mathbf{C}: |\lambda| > r(x) - \varepsilon\}$. As there is a component ω of K with $\inf_{\lambda\in\omega}|\lambda| > \alpha(K) - \varepsilon = r(x) - \varepsilon$, so $U \supseteq \omega$, and hence by Theorem 3, there exists $\delta > 0$ such that $\|y - x\| < \delta$ implies that U contains a component of $\sigma(y)$. Therefore $r(y) > r(x) - \varepsilon$. This proves the lower semicontinuity of r at x. As upper semicontinuity is automatic, it follows that r is continuous at x. Thus we have shown that K is an r-set.

(i) $\Rightarrow$ (iii) by definition.

(iii) $\Rightarrow$ (ii). Assume that (iii) holds and that $\alpha(K) < r(K)$. (We can always construct a normal T with $\sigma(T) = K$, simply by choosing a diagonal operator whose diagonal entries are dense in K.) Now every component ω of K meets the set $\Delta = \{\lambda \in \mathbf{C}: |\lambda| \leqslant \rho\}$ where ρ is chosen such that $\alpha(K) < \rho < r(K)$. Hence by Theorem 1, there is a sequence T_n in $B(H)$ converging to T in norm with $\sigma(T_n) \subseteq \Delta$. Thus $r(T_n) \leqslant \rho$ $(n \geqslant 0)$. But this is impossible by the continuity of r at T. This contradiction shows $\alpha(K) = r(K)$, and so (iii) $\Rightarrow$ (ii). The proof of implication (iii) $\Rightarrow$ (ii) is a simplification of a special case of the proof of Theorem 2.6 in [3].

For $K \in \mathbf{K}$, let $K_0 = \{\lambda \in K$: the component of λ in K is $\{\lambda\}\}$. (Thus $K = K_0$ if and only if K_0 is totally disconnected.) As before, T is a normal operator on H and $\sigma(T) = K$. $\overline{K}_0$ denotes the closure of K_0 in $\mathbf{C}$.

PROPOSITION 2. *The following are equivalent statements*:

(i) K *is a* σ-*set*;

(ii) $K = \bar{K}_0$;

(iii) *for each* $\varepsilon > 0$ *and for each* $\lambda \in K$, $B(\lambda, \varepsilon) = \{\mu \in \mathbf{C}: |\mu - \lambda| < \varepsilon\}$ *contains a component of* K;

(iv) T *is a point of continuity of* σ *in* $B(H)$.

PROOF. That (iii) $\Rightarrow$ (ii) follows from some elementary topology. (This fact is also pointed out in [3, p. 19].)

(ii) $\Rightarrow$ (i). Let A be any Banach algebra, and suppose that $\sigma(x) = K$. Then for each $\varepsilon > 0$, there exist $\lambda_1, \ldots, \lambda_n \in K$ such that $B(\lambda_1, \varepsilon/2) \cup \cdots \cup B(\lambda_n, \varepsilon/2) \supseteq K$. Hence as $\bar{K}_0 = K$, there exist $\mu_i \in K_0$ $(i = 1, \ldots, n)$ such that $|\lambda_i - \mu_i| < \varepsilon/2$. Thus there exist $\delta_i > 0$ $(i = 1, \ldots, n)$ such that $\|y - x\| < \delta_i$ implies that $B(\lambda_i, \varepsilon/2)$ contains a component of $\sigma(y)$ (this follows from Theorem 3, and the fact that $\{\mu_i\}$ is a component of $\sigma(x) = K$). Thus if $\delta = \min_{1 \leqslant i \leqslant n} \delta_i$ then for $\|y - x\| < \delta$ and $\lambda \in \sigma(x)$ we have $|\lambda - \lambda_i| < \varepsilon/2$ for some i, $1 \leqslant i \leqslant n$, and, for each i, $|\lambda_i - \lambda_i'| < \varepsilon/2$ for some λ_i' in $\sigma(y)$, as $B(\lambda_i, \varepsilon/2)$ contains a component of $\sigma(y)$.

Hence $|\lambda - \lambda_i'| < \varepsilon$, or $d(\lambda, \sigma(y)) < \varepsilon$. Thus $\sup_{\lambda \in \sigma(x)} d(\lambda, \sigma(y)) \leqslant \varepsilon$ for $\|y - x\| < \delta$.

But by the upper semicontinuity property, there exists $\delta_0 < \delta$, $\delta_0 > 0$, such that $\|y - x\| < \delta_0$ implies that $\sup_{\lambda \in \sigma(y)} d(\lambda, \sigma(x)) \leqslant \varepsilon$. Thus $\Delta(\sigma(y), \sigma(x)) \leqslant \varepsilon$ for $\|y - x\| < \delta_0$, showing that σ is continuous at x. Hence K is a σ-set. (This argument is a generalization of one due to Newburgh [5] proving that totally disconnected sets are σ-sets.)

(i) $\Rightarrow$ (iv) by definition.

(iv) $\Rightarrow$ (iii). Assume that T is a point of continuity of σ and let $\varepsilon > 0$ and $\lambda \in K = \sigma(T)$. If $B(\lambda, \varepsilon)$ does not contain a component of K, then $S = \mathbf{C} \setminus B(\lambda, \varepsilon)$ is a closed set meeting all the components of $\sigma(T)$, so by Theorem 1, there is a sequence T_n in $B(H)$ with T_n converging to T in norm, but $\sigma(T_n) \subseteq S$ $(n \geqslant 0)$. Hence $\Delta(\sigma(T_n), \sigma(T)) \geqslant d(\lambda, \sigma(T_n)) \geqslant \varepsilon$ $(n \geqslant 0)$. But as σ is continuous at T, $\Delta(\sigma(T_n), \sigma(T)) \to 0$ $(n \to \infty)$. This contradiction shows $B(\lambda, \varepsilon)$ contains a component of $\sigma(T)$, and so (iv) implies (iii). (This result is a simplified proof of a special case of Theorem 3.1 in [3].)

Finally, it is easy to exhibit $K \in \mathbf{K}$ such that $K_0 \neq K = \bar{K}_0$ and to exhibit K such that $\alpha(K) = r(K)$ but no component of K lies on the circle $\{\lambda: |\lambda| = r(K)\}$. (For example take $K = \{1 - 1/n: n \geqslant 1\} \cup \{x + iy: x + y = 1; x, y \geqslant 0\}$.)

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