

NORMAL CLASSES OF PRIME RINGS DETERMINED BY MODULES¹

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ABSTRACT. It is shown that primitive rings with a socle, left weakly primitive rings and prime Johnson rings constitute normal classes of rings. The technique involves characterizing each class as all rings with a faithful module of a certain type.

1. Introduction. A Morita context (R, V, W, S) consists of two rings R and S and two bimodules ${}_R V_S$ and ${}_S W_R$ together with mappings $V \times W \rightarrow R$ and $W \times V \rightarrow S$ (written multiplicatively) which induce bimodule homomorphisms $V \otimes_S W \rightarrow R$ and $W \otimes_R V \rightarrow S$ and which satisfy the associativity conditions

$$(vw)v' = v(wv') \quad \text{and} \quad (wv)w' = w(vw')$$

for all v, v' in V and w, w' in W . See Amitsur [1] for details. An equivalent formulation is that $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is a ring with the usual matrix operations. An important example is the *standard context* (R, M, M^*, E) determined by an R -module M where $E = \text{end}_R M$. We call a Morita context (R, V, W, S) *S-faithful* if $S \neq 0$ and $VsW = 0$ implies $s = 0$ for $s \in S$. Then a *normal class* \mathcal{P} is defined [3] to be a class of prime rings such that if (R, V, W, S) is an S -faithful context with $R \in \mathcal{P}$ then necessarily $S \in \mathcal{P}$. These normal classes enjoy many pleasant properties and the reader is referred to [3] for details. Examples of normal classes include the classes of prime and left primitive rings [1] and it is the intention here to show that the class of primitive rings with a nonzero socle is normal as is the class of prime nonsingular rings with uniform left ideal. Rowen [5] calls these *prime Johnson rings* and has characterized them as essential prime subrings of primitive rings with nonzero socle.

As well as these, two more classes of rings are of interest to us here. Zelmanowitz [7] has called a ring *left weakly primitive* if it has a faithful, monofrom, compressible left module and has proved a version of the Jacobson Density Theorem for these rings. Finally, Ortiz [4] has introduced the concept of a K -primitive ring which we will redefine later.

The principal aim of this note is to prove the following

THEOREM. *The following classes of prime rings are normal:*

- (1) *the primitive rings with nonzero socle,*

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- (2) the left weakly primitive rings,
- (3) the prime Johnson rings,
- (4) the K -primitive rings.

2. Module properties and Morita contexts. Throughout this paper, unless otherwise mentioned, rings need not have a unity, modules are left modules, and homomorphisms are written on the right. If R is a ring, then R^1 denotes R if R has a unity and, if not, it denotes the usual ring with unity containing R as an ideal. If M is a module and $X \subseteq M$ the left annihilator of X will be written $l(X)$. All classes of rings and modules are assumed to be nonempty and closed under isomorphisms.

The classes of modules to be considered consist of prime modules. Here a module M is called *prime* if $RM \neq 0$ and, whenever $Am = 0$ where $m \in M$ and A is an ideal of R , then either $AM = 0$ or $m = 0$. Another notion of prime module was introduced by Zelmanowitz [6] and, to distinguish this from the above concept, we refer to these as standard prime. Thus M is *standard prime* if $M \neq 0$ and $m_1 M^* m_2 = 0$, $m_i \in M$, implies $m_1 = 0$ or $m_2 = 0$ where $M^* = \text{Hom}_R(M, R)$. Observe that a ring R is prime if and only if ${}_R R$ is prime and faithful if and only if ${}_R R$ is standard prime and faithful.

A module M is called *compressible* if $RM \neq 0$ and M can be embedded in each nonzero submodule. Note that all compressible and standard prime modules are prime and that each of these classes of modules is closed under taking nonzero submodules.

A module M is called *monoform* if every partial endomorphism $\alpha: N \rightarrow M$, $N \subseteq M$, is either zero or monic. Such modules are uniform and we shall use the fact that a uniform, nonsingular module is necessarily monoform. We use the term *nonsingular* to mean that the singular submodule $Z(M)$ is zero. Here $Z(M) = \{m \in M \mid l(m) \text{ is essential in } R\}$.

We shall define a module ${}_R M$ to have *property K* if M is prime and $l(M/N) = l(M)$ for ${}_R N \subseteq {}_R M$ only when $N = 0$. Then a *K-primitive ring* [4] is a ring with a faithful module having property K .

PROPOSITION. *Let (R, V, W, S) be a Morita context and let $M = R^1 m$ be a left R -module such that $(VSW)M \neq 0$. Put*

$$U = \{w \in W \mid Vwm = 0\}.$$

If ${}_R M$ has any of the following properties, so also does the S -module W/U :

- (1) irreducible,
- (2) prime,
- (3) compressible,
- (4) prime and monoform.

If in addition (R, V, W, S) is S -faithful we can add:

- (5) faithful,
- (6) standard prime,
- (7) property K .

PROOF. (1) and (2) are proved as in Amitsur [1] and (5) is an easy consequence of the definitions. The proof of (3) is straightforward and is left to the reader.

(4) W/U is prime by (2). Suppose $0 \neq \alpha: W_0/U \rightarrow W/U$ is S -linear where W_0 is an S -submodule of W . We must show α is monic. Let $(w_0 + U)\alpha = w_1 + U \neq 0$ and define $\alpha_1: Vw_0m \rightarrow M$ by $(vw_0m)\alpha_1 = vw_1m$. If it happens that $vw_0m = 0$, then $VWvw_1m = 0$ so $vw_1m = 0$ because M is prime. Hence α_1 is well defined and nonzero so it is monic by hypothesis. We show that this implies α is monic. Suppose, if possible, that $(w_2 + U)\alpha = 0$ where $w_2 \in W_0 - U$. Then $Vw_2m \neq 0$ so, since M is uniform (being moniform), we have $Vw_2m \cap Vw_0m \neq 0$, say $v_2w_2m = v_0w_0m \neq 0$. Then $W(v_2w_2 - v_0w_0) \subseteq U$ so, for all $w \in W$,

$$0 = vw_2(w_2 + U)\alpha = (vw_0w_0 + U)\alpha = vw_0w_1 + U.$$

This means $VWv_0w_1m = 0$ so $0 = v_0w_1m = (v_0w_0m)\alpha_1$, a contradiction. Hence α is monic as required.

(6) Assume M is standard prime. If $w_1 + U$ and $w_2 + U$ are nonzero in W/U , let $v_1w_1m \neq 0 \neq v_2w_2m$, $v_i \in V$. By hypothesis there exists $\lambda: M \rightarrow R$ such that $[(v_1w_1m)\lambda](v_2w_2m) \neq 0$. Define $\gamma: W/U \rightarrow S$ by $(w + U)\gamma = w(m\lambda)v_2$. If $w \in U$ then $Vwm = 0$ so $Vw(m\lambda)v_2W = 0$. This implies $w(m\lambda)v_2 = 0$ (since the context is S -faithful) so γ is well defined. It remains to show that $(w_1 + U)\gamma(w_2 + U) \neq 0$. But if this equals 0 then $Vw_1(m\lambda)v_2w_2m = 0$, contrary to the choice of λ .

(7) Let $W \supseteq {}_S T \supset U$ and suppose that $l(W/T) = l(W/U)$. Pick $t \in T - U$. Then $Vtm \neq 0$ and $\exists a \in R$ such that $0 \neq aM \subseteq Vtm$ by property K . Since M is prime $l(M)$ is a prime ideal of R so $a \notin l(M)$ and $VW \not\subseteq l(M)$ imply that $VWaVW \not\subseteq l(M)$ and so $\exists b \in WaV$ with $VbW \not\subseteq l(M)$. Hence $VbWm \neq 0$, $bW \not\subseteq U$.

On the other hand $aVWM \subseteq aM \subseteq Vtm$, so given $v \in V$, $w \in W \exists v' \in V$ with $avwm = v'tm$. Hence $VW(avw - v't)m = 0$, $W(avw - v't) \subseteq U$, and $Wavw \subseteq T$. Thus $WaV \subseteq l(W/T) = l(W/U)$ so $WaVW \subseteq U$. In particular $bW \subseteq U$ giving a contradiction. Therefore W/U has property K . \square

3. Proof of Theorem. In each of the first three cases we characterize the rings considered as those having a faithful module of some specific type.

(1) If R is primitive with nonzero socle and L is a minimal left ideal of R , then L is a faithful, standard prime, irreducible R -module. Conversely, if M is a faithful, standard prime, irreducible module and $0 \neq \alpha \in M^*$, then $M \cong M\alpha$ which is a minimal left ideal of the prime ring R . Hence R is primitive with nonzero socle. The normality of this class now follows from (1), (5) and (6) of the Proposition.

(2) Since compressible modules are prime, this is immediate from the definitions and parts (3), (4) and (5) of the Proposition.

(3) If R is a prime Johnson ring and L is a nonsingular uniform left ideal of R , then L is a faithful, standard prime, moniform module. For the converse suppose that M is a faithful, standard prime, moniform module. Let $0 \neq \alpha \in M^*$. Then $(M\alpha)^2 \neq 0$ so let $m'am \neq 0$ where $m', m \in M$. The map $am: M \rightarrow M$ is monic by hypothesis so $M \cong M\alpha \subseteq R$. Suppose now that $b \in Z(M\alpha)$. Then $M\alpha \cap l(b) \neq 0$ so the map $\beta: x \mapsto xb$ is not monic on $M\alpha$. Hence $\beta = 0$ so $M\alpha \subseteq l(b)$. But then $M\alpha Z(M\alpha) = 0$ whence $Z(M\alpha)^2 = 0$. Thus $Z(M\alpha) = 0$ since R is a prime ring. Using this characterization of prime Johnson rings the normality follows from (4), (5) and (6) of the Proposition.

(4) Let ${}_R M$ have property K . We are done if the property K is inherited by nonzero submodules, for then we can take $M = R^1 m$ and the normality follows from (5) and (7) of the Proposition. Suppose $0 \neq {}_R N \subseteq M$. Then $l(N) = l(M)$. If ${}_R K \subseteq N$ and $l(N/K) = l(N)$, then $l(M/K) \subseteq l(N/K) = l(N) = l(M) \subseteq l(M/K)$, so $K = 0$ and N has property K as required. \square

Bergman has given an example of a ring primitive on one side but not on the other but the analogous question for weakly primitive rings is still open. Of course the notion of a primitive ring with a socle is left-right symmetric but the following example shows that this is not true of the prime Johnson rings.

EXAMPLE. Let F be a field with a monomorphism $\alpha: F \rightarrow F$ which is not onto. Then the skew polynomial ring $R = F[x, \alpha]$ is prime, left and right nonsingular, and has a uniform left ideal but no uniform right ideal. In fact, the prime and nonsingularity conditions are clear since R is a domain, and ${}_R R$ is uniform since R is left Öre. However, if T is a uniform right ideal and $0 \neq b \in T$, the map $r \mapsto br$ is an embedding $R \rightarrow T$. This implies R is right Öre, a contradiction. \square

REMARK. If \mathcal{P} is any normal class of prime rings, we may define a radical in the category of rings by

$$N_{\mathcal{P}}(R) = \bigcap \{P \mid P \text{ is an ideal of } R \text{ and } R/P \in \mathcal{P}\}.$$

Then $N_{\mathcal{P}}$ is a normal and special radical (see [3]) and was considered in [1] when \mathcal{P} is the class of left primitive rings or the class of prime rings. If \mathcal{P} consists of the left weakly primitive rings then $N_{\mathcal{P}}$ is the weak radical [2] while if \mathcal{P} is the class of K -primitive rings, the radical is discussed in [4]. Properties of these radicals (e.g. $N_{\mathcal{P}}[M_n(R)] = M_n[N_{\mathcal{P}}(R)]$; see [2] and [4]) now follow easily from general results in [3].

Call a left ideal L in a ring R full if $rR \subseteq L$, $r \in R$, implies $r \in L$. Suppose \mathcal{N} is a class of prime modules giving rise, as in the theorem, to a normal class

$$\mathcal{P} = \{R \mid R \text{ has a faithful module in } \mathcal{N}\}.$$

Then the radical $N_{\mathcal{P}}$ above has the following description

$$N_{\mathcal{P}}(R) = \bigcap \{L \mid L \text{ is a full left ideal of } R \text{ and } R/L \in \mathcal{N}\}.$$

The verification is omitted.

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