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NORMING NIL ALGEBRAS

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ABSTRACT. A commutative nil algebra with countable basis is normable, but a commutative nilpotent algebra is not necessarily normable.

Let A be a commutative algebra over the complex field, $\mathbf{C}$. We say that A is *normable* if there is an algebra norm on A, or, equivalently, if A can be embedded in a normed algebra. A deep recent theorem of Esterle [3] shows that, if the continuum hypothesis holds, then each integral domain without identity, of cardinality at most that of the continuum is normable. For a discussion of this result, and of related results, see [2, §9]. We consider here an opposite extreme, the normability of nil algebras.

Throughout, we consider only commutative, linear associative algebras over $\mathbf{C}$. Let A be an algebra. If $n \in \mathbf{N}$, we write A^n for the linear span of the set of products of n elements of A. An element a of A is *nil* if $a^n = 0$ for some $n \in \mathbf{N}$, the algebra A is *nilpotent* if $A^n = 0$ for some $n \in \mathbf{N}$, and A is *nil* if each element of A is nilpotent. A vector space norm, $\|.\|$, on an algebra A is *submultiplicative*, or an *algebra norm*, if $\|ab\| \leqslant \|a\|\,\|b\|$ $(a, b \in A)$.

First, suppose that A is a nilpotent algebra.

If $A^2 = 0$, then certainly A is normable. For let $\{a_\alpha\}$ be a basis for A over $\mathbf{C}$, and set $\|\Sigma_1^n \lambda_i a_{\alpha_i}\| = \Sigma_1^n |\lambda_i|$. Then $\|.\|$ is a vector space norm on A, and in this case each vector space norm is an algebra norm.

Now consider the case that $A^3 = 0$. An example to show that such an A may not be normable is already implicit in the note [4] of Esterle. If $p > 1$, let l^p denote the usual Banach space. Then l^p is a commutative Banach algebra with respect to coordinatewise multiplication, and l^q is an ideal in l^p if $1 < q < p$. Let $A = l^p/l^q$. Then, by [4, Theorem 3.1], A is normable if and only if $p < 2q$. By taking p, q so that $3q \geqslant p > 2q \geqslant 2$, we obtain an algebra A with $A^3 = 0$ which is not normable. However, this example uses the "main boundedness theorem" of Bade and Curtis [1], and it may be of interest to give a totally elementary example.

EXAMPLE 1. *There exists a commutative algebra A with $A^3 = 0$ which is not normable.*

PROOF. As a vector space, A has as basis the set $\{e_s: s \in [0, 1)\}$. The multiplication is given by

$$e_s e_t = \begin{cases} e_0/|s - t| & (s, t \in (0, 1), s \neq t), \\ 0 & (\text{otherwise}). \end{cases}$$

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It is clear that the product of any three elements of A is zero, so that A is associative and $A^3 = 0$.

Now suppose that $\|.\|$ is an algebra norm on A. Then

$$0 < \|e_0\| \leqslant |s - t| \, \|e_s\| \, \|e_t\| \qquad (s, t \in (0, 1), s \neq t).$$

For $n \in \mathbf{N}$, let $U_n = \{s \in (0, 1): \|e_s\| > n\}$. If $s \in (0, 1)$ and if $0 < |t - s| < \|e_0\|/n^2\|e_s\|$, then $t \in U_n$. Thus, each U_n is open and dense in (0, 1). By Baire's theorem, $\bigcap U_n \neq \varnothing$, a contradiction. Thus A is not normable.

The above example clearly uses the uncountability of the dimension of A. In fact, we shall now show that each nil, and in particular each nilpotent, algebra of countable dimension is normable. (A proof, shorter than the following, that each nilpotent algebra of countable dimension is normable can easily be given.) The result is an easy consequence of the following lemma.

LEMMA 2. *Let B be a finite-dimensional algebra, and let A be a nil subalgebra such that B is algebraically generated by A and by an element b_0 of B with $b_0^2 \in A$. Suppose that there is an algebra norm on A. Then there is an algebra norm on B extending the norm on A.*

PROOF. Let $a_1, \ldots, a_p$ be a basis of A. Then $\{b_0 + A, a_1 b_0 + A, \ldots, a_p b_0 + A\}$ spans the vector space B/A. Choose a linearly independent subset of this set, say it is $\{z_1 + A, \ldots, z_k + A\}$, where $z_1, \ldots, z_k \in B$. Then each $b \in B$ has a unique expression in the form $b = a + \sum_1^n \alpha_i z_i$, with $a \in A$ and $\alpha_1, \ldots, \alpha_k \in \mathbf{C}$: we set $\pi(b) = (\alpha_1, \ldots, \alpha_k)$.

Note that

$$z_i z_j \in A \qquad (i, j = 1, \ldots, k) \tag{1}$$

because $(a_r b_0)(a_s b_0) \in A$ for $r, s = 1, \ldots, p$.

For $a \in A$, let $T_a: \mathbf{C}^k \to \mathbf{C}^k$ be defined by

$$T_a(\lambda_1, \ldots, \lambda_k) = \pi[a(\lambda_1 z_1 + \cdots + \lambda_k z_k)].$$

Then T_a is a linear map on $\mathbf{C}^k$, and we regard T_a as an element of $M_k(\mathbf{C})$. The map $a \mapsto T_a$ of A into $M_k(\mathbf{C})$ is an algebra homomorphism, and so $\{T_a: a \in A\}$ is a set of commuting matrices in $M_k(\mathbf{C})$. Thus, by [5, p. 134], we can choose a basis of $\mathbf{C}^k$ so that each matrix T_a is lower triangular with respect to the new basis. By replacing each z_i by the appropriate linear combination of $z_1, \ldots, z_k$, we can suppose that each T_a already has a lower triangular matrix. Note that equation (1) still holds for the new $z_1, \ldots, z_k$. Let $T_a = [m_{ij}(a)]$ for $a \in A$, where each m_{ij} is a linear functional on A and $m_{ij} = 0$ if $j > i$. Clearly, each m_{ii} is a homomorphism $A \to \mathbf{C}$.

Take $a \in A$ and let $\phi: A \to \mathbf{C}$ be a homomorphism. If $\phi(a) \neq 0$, let $b = \phi(a)^{-1}a$. Since A is a nil algebra, there exists $c \in A$ with $b + c = bc$, so that $1 + \phi(c) = \phi(c)$, a contradiction. Thus $\phi(a) = 0$. This shows that we can suppose that $T_a = [m_{ij}(a)]$ for $a \in A$, where each m_{ij} is a linear functional on A and $m_{ij} = 0$ if $j \geqslant i$.

For $j \in \{1, \ldots, k\}$, let $L_j = \mathrm{lin}\{A, z_1, \ldots, z_j\}$, and take $L_0 = A$. Then we have shown that, with our choice of $z_1, \ldots, z_k$, we have $Az_j \subset L_{j-1}$ $(j = 1, \ldots, k)$.

Using (1), it follows that

$$(2) \qquad Bz_j \subset L_{j-1} \qquad (j = 1, \ldots, k).$$

We shall successively define $\|z_1\|, \ldots, \|z_k\|$ (with $\|z_j\| > 0$). When we have defined $\|z_1\|, \ldots, \|z_j\|$ we extend the norm to L_j by setting

$$\|a + \alpha_1 z_1 + \cdots + \alpha_j z_j\| = \|a\| + |\alpha_1|\,\|z_1\| + \cdots + |\alpha_j|\,\|z_j\|.$$

Clearly, we obtain on each L_j a norm which extends the norm on A. We shall show that, with our choice of the $\|z_j\|$, the norm on $L_k = B$ is submultiplicative.

In fact, we choose $\|z_i\|$ so that for $i = 1, \ldots, k$,

$$(3) \qquad \|az_i\| \leqslant \|a\|\,\|z_i\| \quad (a \in L_{i-1}) \quad \text{and} \quad \|z_i^2\| \leqslant \|z_i\|^2.$$

Suppose that $j = 1$, or that $j > 1$ and that $\|z_1\|, \ldots, \|z_{j-1}\|$ have been chosen so that (3) holds for $i = 1, \ldots, j-1$. If $a \in L_{j-1}$, then $az_j \in L_{j-1}$ by (2), and so $\|az_j\|$ and $\|z_j^2\|$ have already been defined. Define a norm on L_{j-1}^2 by setting $\|(x, y)\| = \|x\| + \|y\|$, and let $X_j = \{(a, b) \in L_{j-1}^2 \colon az_j = b\}$. Then X_j is a linear subspace of the finite-dimensional normed space L_{j-1}^2, and hence X_j is closed. Let $Y_j = \{(a, b) \in X_j \colon \|b\| = 1\}$, a closed subset of L_{j-1}^2. The map $(a, b) \mapsto \|a\|$ is a continuous function on L_{j-1}^2, and clearly $\delta_j \equiv \inf\{\|a\| \colon (a, b) \in Y_j\}$ is attained. If $(a, b) \in Y_j$, then $a \neq 0$, and so $\delta_j > 0$. We take $\|z_j\| = \max\{\delta_j^{-1}, \|z_j^2\|^{1/2}\}$, and it is then easy to check that (3) holds in the case that $i = j$. Thus, we can choose $\|z_1\|, \ldots, \|z_k\|$ so that (3) holds for $i = 1, \ldots, k$.

We now confirm that the norm is indeed submultiplicative on B. Take $a, b \in B$, say $a = a_0 + \sum_1^k \alpha_i z_i$, $b = b_0 + \sum_1^k \beta_i z_i$, where $a_0, b_0 \in A$, and $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k \in \mathbf{C}$. Then $ab = a_0 b_0 + \sum_1^k c_j z_j + \sum_1^k \alpha_j \beta_j z_j^2$, where

$$c_j = \alpha_j b_0 + \beta_j a_0 + \sum_{i=1}^{j-1} (\alpha_i \beta_j + \alpha_j \beta_i) z_i,$$

so that $c_j \in L_{j-1}$. Then

$$\|ab\| \leqslant \|a_0 b_0\| + \sum_1^k \|c_j z_j\| + \sum_1^k |\alpha_j \beta_j|\,\|z_j^2\|.$$

But $\|a_0 b_0\| \leqslant \|a_0\|\,\|b_0\|$ because $\|.\|$ is an algebra norm on A, and $\|c_j z_j\| \leqslant \|c_j\|\,\|z_j\|$, $\|z_j^2\| \leqslant \|z_j\|^2$ by (3). Thus

$$\begin{aligned}\|ab\| &\leqslant \|a_0\|\,\|b_0\| + \|b_0\| \sum |\alpha_j|\,\|z_j\| + \|a_0\| \sum |\beta_j|\,\|z_0\| \\ &\quad + \sum_{i=1}^{j-1} \big(|\alpha_i|\,|\beta_j| + |\alpha_j|\,|\beta_i|\big)\|z_i\|\,\|z_j\| + \sum_1^k |\alpha_j|\,|\beta_j|\,\|z_j\|^2 \\ &\leqslant \Big(\|a_0\| + \sum |\alpha_j|\,\|z_j\|\Big)\Big(\|b_0\| + \sum |\beta_j|\,\|z_j\|\Big) \\ &= \|a\|\,\|b\|,\end{aligned}$$

as required.

We now give our result.

THEOREM 3. *Let A be a nil algebra with countable basis. Then A is normable.*

PROOF. We can write $A = \bigcup_{n=0}^{\infty} A_n$, where $A_0 = \{0\}$, where each A_n is a finite-dimensional subalgebra of A_{n+1}, and where A_{n+1} is the algebra generated by A_n and by an element a_{n+1} of A_{n+1} with $a_{n+1}^2 \in A_n$. The algebras A_n are nil, and so we can successively extend an algebra norm from A_n to A_{n+1}. Hence, we can construct an algebra norm on A.

A result which is apparently more general is also true.

THEOREM 4. *Let A be a radical algebra with countable basis. Then A is normable.*

However, the extra generality is illusory because a theorem of Amitsur (see [**6**, p. 20]) shows that a radical algebra with countable basis is, in fact, a nil algebra.

It is not true that every algebra with countable basis can be normed.

EXAMPLE 5. *There exists a commutative algebra A with a countable basis which is not normable.*

PROOF. Let P be the free polynomial algebra in the countable family of variables $(X_n\colon n \in \mathbf{N})$, and let P_0 denote the polynomials with zero constant term. Let I be the ideal in P_0 generated by the elements $X_1X_n - nX_n$ $(n \geq 2)$, and let $A = P_0/I$. Then A is a commutative algebra with a countable basis, and we shall show that A is not normable.

First note that $X_n \notin I$ for $n \geqslant 2$. For if so, there exists $n \geq 2$, $k \in \mathbf{N}$, and $p_2, \ldots, p_k \in P_0$ with $X_n = \sum_{j=2}^{k}(X_1X_j - jX_j)p_j$. Take $X_1 = X$, $X_n = Y$, and $X_j = 0$ for $j \in \mathbf{N} \setminus \{1, n\}$. Then $Y = (XY - nY)p(X, Y)$ for a polynomial p, and so $1 = (X - n1)p(X, Y)$, which is not possible. Thus $X_n \notin I$ for $n \geq 2$, as required.

Suppose that $\|.\|$ is an algebra norm on A. Set $a_n = X_n + I \in A$ for $n \in \mathbf{N}$. Then $a_1a_n = na_n$ $(n \geqslant 2)$, and so $n\|a_n\| \leq \|a_1\|\,\|a_n\|$ $(n \geq 2)$. Since $X_n \notin I$, $\|a_n\| \neq 0$ for $n \geqslant 2$. Thus $n \leqslant \|a_1\|$ $(n \geqslant 2)$, a contradiction. Hence A is not normable.

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