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k-DISCRETE DIFFERENTIALS OF CERTAIN OPERATORS ON BANACH SPACES

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ABSTRACT. By observing a convex property of discrete differences, one-sided k-discrete, k-discrete Gâteaux and k-discrete Fréchet differentials are introduced. It is proved that a locally bounded n-convex function has k-discrete Fréchet differentials for $1 \leq k \leq n-2$ and one-sided $(n-1)$-discrete differentials at every point of its domain. Various properties of discrete differentials of an n-convex function are studied. As an application of these results the author proves that an n-convex function has a strong $(n-2)$-Taylor series expansion and an $(n-1)$th Fréchet differential provided it has a strong n-Taylor series expansion about the point.

1. Introduction. Higher differentials of a function on a Banach space can be defined either inductively with reference to the lower differentials or directly with no reference to lower order differentials. Riemann (see Butzer and Kozakiewez **[2]**), Peano (H,β)-Peano (see Weil **[13]**, Ash **[1]**), and Taylor differentials (see Nashed **[11]**) are introduced in literature to define higher differentiability directly without reference to lower ones. One-sided k-discrete, k-discrete Gâteaux and k-discrete Fréchet differentials are introduced as direct differentials by using the discrete difference notion (see Dayal **[5]**), which is the extension to vector-valued functions of the divided differences of numerical analysis. The notion also induces a class of functions called n-convex functions, given by the author in 1972 in **[4]**, which was introduced in literature by various approaches.

The class of n-convex functions contains the class of subconvex functions commonly known as convex functions $(n = 2)$, the monotonic functions $(n = 1)$, and the class of positive-valued functions. The first main result (Theorem 3.2) states that the k-discrete differential for $1 \leq k \leq n-2$ and one-sided $(n-1)$-discrete differential of a locally bounded n-convex function exist at every point of its domain. The second main result (Theorem 3.4) states that the one-sided k-discrete differential of an n-convex function satisfies a kind of uniform continuity (see Definition 3.3) for $1 \leq k \leq n-2$. These results together with Theorem 3.5 and a local representation theorem (see Theorem 3.2 in **[5]**) are used by the author to prove that n-convex functions which are locally bounded admit a strong $(n-2)$ Taylor series expansion (for definition, see **[5]**), which, in turn, proves the existence of the $(n-1)$th Fréchet differential of the function. This result was proved by the author in 1972 **[4]** and the detailed proof will be given elsewhere.

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2. Preliminaries. The meaning of differentiability and higher differentiability in the inductive sense is elegantly presented by Dieudonné [7] (see also [5]). The notions of one-sided Gâteaux, Gâteaux, and Fréchet differentiabilities along with the various direct differentials can be found in Nashed [12] (see also [4], [5]). We give the notion of discrete differences, which introduces the concept of one-sided k-discrete, k-discrete Gâteaux and k-discrete Fréchet differentials.

An n-discrete difference $[\Delta_n h](t_0, \ldots, t_n)$ of a function h: $[a, b] \to F$, where $(t_0, \ldots, t_n)$ is a finite sequence of distinct numbers in the interval $[a,b]$ is the coefficient of t^n in the unique polynomial $P(t)$ of degree $\leqslant n$ such that $P(t_i) = h(t_i)$ for $i = 0, 1, 2, \ldots, n$. Thus

$$[\Delta_n h](t_0, \ldots, t_n) = \sum_{k=0}^{n} \left[1 \Big/ \left(\prod_{\substack{j \in [0,n] \\ j \neq k}} (t_k - t_j) \right) \right] \cdot h(t_k) \tag{2.1}$$

and it is shown in [5] that the discrete difference satisfies a kind of convex property, namely, for $t_0 < t < t_n$ with $t_i \neq t_j$ for $i \neq j$ and $t \neq t_j$, $j = 1, 2, \ldots, n$,

$$\begin{aligned} [\Delta_n h](t_0, \ldots, t_n) = {} & \frac{(t - t_0)}{(t_n - t_0)} \cdot [\Delta_n h](t_0, \ldots, t_{n-1}, t) \\ & + \left(1 - \frac{(t - t_0)}{(t_n - t_0)} \right) [\Delta_n h](t_1, \ldots, t_n, t). \end{aligned}$$

Let E and F be two Banach spaces and A be an open subset of E. Let f: $A \to F$. For every $y \in A$ and $v \in E$ we define a function $[h(y, v)]t = f(y + tv)$. Let $[\Delta_k h(y, v)](t_0, \ldots, t_k)$ be the k-discrete difference of $h(y, v)$ defined for any finite sequence $t = (t_0, \ldots, t_k)$ with $t_i \neq t_j$, $i \neq j$, and such that $y + tv$ is in the domain of f for all $j = 0, 1, \ldots, k$, $y \in A$ and $v \in E$. We write

$$[\Delta_k h(y, v)](t_0, \ldots, t_k) = [\Delta_k h(y, v)]t.$$

DEFINITION 2.1. Let E and F be Banach spaces and A be an open set of E. For a function f: $A \to F$ and some $y \in A$ suppose that the limit

$$(\hat{f}^{(k)}y)v = (k!) \lim_{t \to 0+} [\Delta_k h(y, v)]t \tag{2.2}$$

exists for all $v \in E$ in the sense that, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t = (t_0, \ldots, t_k)$ with $0 \leqslant t_i < \delta$; $t_i \neq t_j$, $i \neq j$, $i, j = 0, 1, \ldots, k$,

$$\|(k!)[\Delta_k h(y, v)]t - (\hat{f}^{(k)}y)v\| < \varepsilon. \tag{2.3}$$

Then $(\hat{f}^{(k)}y)$ is called the *one-sided k-discrete differential* of f at y. A function f: $A \to F$ is said to have a *k-discrete Gâteaux differential* $(f^{(k)}y)$: $E^k \to F$ if $(f^{(k)}y)$ is a k-linear operator such that

$$(\hat{f}^{(k)}y)u = (f^{(k)}y)(u, u, \ldots, u),$$

provided the limit (2.2) exists.

In addition, if the limit (2.2) is uniform for vectors v such that $\|v\| \leqslant 1$, then $(f^{(k)}y)$ is called the *k-discrete Fréchet differential*. 1-discrete Gâteaux and 1-discrete bounded Fréchet differentials are equivalent to Gâteaux and Fréchet differentials, respectively. Further, the existence of the k-discrete Gâteaux or k-discrete Fréchet differential does not imply the existence of the corresponding $(k-1)$-discrete differential.

3. Discrete differentials of an n-convex function. Discrete differences are used to define a class of functions called n-convex functions. R. Ger ([9], [10]) has defined these by using the finite distinct sequence with equal spacing, whereas we use the arbitrary finite distinct sequence. It is proved that the k-discrete Fréchet differential for $1 \leqslant k \leqslant n-2$ and the one-sided $(n-1)$-discrete differential of a locally bounded n-convex function exist at every point of its domain.

DEFINITION 3.1. Let A be an open subset of a Banach space E. The k-discrete difference $\Delta_k h(y, v)$ associated with f: $A \to \mathbf{R}$ is said to be *monotonically increasing* if for every $y \in A$ and $v \in E$,

$$[\Delta_k h(y, v)]t \leqslant [\Delta_k h(y, v)]t^*$$

for any sequences t, t^* such that $y + t_i v$ and $y + t_i^* v$ are in A for $i = 0, 1, \dots, k$, and $t \leqslant t^*$ in the sense that $t_i \leqslant t_i^*$ for $i = 0, 1, \dots, k$.

DEFINITION 3.2. Let A be an open subset of a Banach space E. The k-discrete difference $\Delta_k h(y, v)$ associated with f: $A \to \mathbf{R}$ is said to be *uniformly bounded* in the δ-neighbourhood N_δ of $y_0 \in A$ if there exists a $\delta' > 0$ and a constant M such that whenever $|t_i| < \delta', y \in N_\delta$ and $\|v\| \leqslant 1$, then $y + t_i v \in A$ and

$$|[\Delta_k h(y, v)]t| \leqslant M.$$

DEFINITION 3.3. Let A be an open subset of a Banach space E. The k-discrete difference $\Delta_k h(y, v)$ associated with f: $A \to \mathbf{R}$ is said to be *strongly uniformly continuous* in the $\delta/2$-neighbourhood $N_{\delta/2}$ of $y_0 \in A$ if, for a given $\varepsilon > 0$, there is a $\delta'' > 0$ such that whenever $y \in N_{\delta/2}$, $|s_i| < \delta/4$, $\|v\| \leqslant 1$, $|t_i| < \delta/4$ and $\sup_{0 \leqslant i \leqslant k} |t_i - s_i| < \delta''$ then

$$|[\Delta_k h(y, v)]t - [\Delta_k h(y, v)]s| < \varepsilon.$$

DEFINITION 3.4. Let A be an open set of a Banach space E. A function f: $A \to \mathbf{R}$ is said to be *n-convex* if for all $y \in A$ and $v \in E$, the function $\Delta_n h(y, v)$ does not change sign as a function . f $t = (t_0, \dots, t_n)$.

THEOREM 3.1. *Let A be an open subset of a Banach space E and f: $A \to$* $\mathbf{R}$.

(a) *If f is n-convex then $\Delta_{n-1}h(y, v)$ is monotonic.*

(b) *If f is n-convex and bounded in a neighborhood N_δ of y_0 then*:

(i) $\Delta_{n-1}h(y, v)$ *is uniformly bounded in $N_{\delta/2}$,*

(ii) $\Delta_k h(y, v)$ *is strongly uniformly continuous in $N_{\delta/2}$,* $0 \leqslant k \leqslant n-2$.

PROOF. Let $s \leqslant t$, $s = (s_0, \dots, s_{n-1})$, $t = (t_0, \dots, t_{n-1})$ and $\tau^i = (s_0, \dots, s_{i-1}, t_i, \dots, t_{n-1})$. Then $\tau^0 = t$ and $\tau^n = s$. We may assume that $[\Delta_n h(y, v)]t \geqslant 0$ for a particular y and v that we are considering. Thus

$$\begin{aligned}&[\Delta_{n-1}h(y, v)]\tau^i - [\Delta_{n-1}h(y, v)]\tau^{i-1}\\ &\qquad = (s_i - t_i)[\Delta_n h(y, v)](s_0, \dots, s_i, t_i, \dots, t_{n-1}) \geqslant 0 \quad \text{for all } i, 0 \leqslant i \leqslant n.\end{aligned}$$

If $t_i = s_i$, the right side is not defined, but the left side is trivially 0. This proves (a).

(b)(i) From (2.1) we immediately see that the specific values

$$[\Delta_{n-1}h(y, v)](-(\delta/4)(1 + 1/(n-1)), \ldots, (\delta/4)(1+1))$$

and

$$[\Delta_{n-1}h(y, v)]((\delta/4)(1 + 1/(n-1)), \ldots, (\delta/4)(1+1))$$

are uniformly bounded (by, say, B_{n-1}) if $y \in N_{\delta/2}$ and $\|v\| \leqslant 1$. Thus, by monotonicity, if $|t_i| < \delta/2$, $[\Delta_{n-1}h(y, v)]t$ is bounded by the same number.

(b)(ii) By downward induction on k, we prove that $[\Delta_k h(y, v)]t$ is uniformly continuous and so uniformly bounded for $y \in N_{\delta/2}$, $\|v\| \leqslant 1$ and $|t_j| < \delta/4$, $j = 0, 1, \ldots, k$. Now

$$\begin{aligned}
&[\Delta_k h(y, v)](t_0, \ldots, t_{i-1}, t', t_{i+1}, \ldots, t_k)\\
&\qquad - [\Delta_k h(y, v)](t_0, \ldots, t_{i-1}, t'', t_{i+1}, \ldots, t_k)\\
&\qquad = (t' - t'')[\Delta_{k+1}h(y, v)](t_0, \ldots, t_{i-1}, t', t'', t_{i+1}, \ldots, t_k).
\end{aligned}$$

We get a small change in values of $\Delta_k h(y, v)$ when only one coordinate t changes by a small number. A general uniform continuity involves a sum of $(k-1)$ such differences.

As an immediate consequence we get the following theorem.

THEOREM 3.2. *If a function f: $A \to \mathbf{R}$ is n-convex ($n \geqslant 2$) and locally bounded where A is any open subset of a Banach space E, then*

(i) *For $0 \leqslant k \leqslant n-2$, the k-discrete differential $(\hat{f}^{(k)}y)$ exists and is bounded for every $y \in A$.*

(ii) *The one-sided $(n-1)$-discrete differential $(\hat{f}^{(n-1)}y)$ exists and is bounded for every $y \in A$.*

(iii) *f is uniformly continuous in a neighbourhood of every point $y \in A$.*

THEOREM 3.3. *Let A be an open subset of a Banach space E. Let f: $A \to \mathbf{R}$ be n-convex, $n \geqslant 2$, and bounded in N_δ, the δ-neighbourhood of $y_0 \in A$. Then for every $\varepsilon > 0$ and every sequence $t = (t_0, \ldots, t_k)$ of distinct numbers such that $|t_i| < \delta/4$, there is a number $\delta' > 0$ such that*

$$|[\Delta_k h(y', v)]t - [\Delta_k h(y'', v)]t| < \varepsilon$$

whenever $\|y' - y_0\| < \delta/4$, $\|y'' - y_0\| < \delta/4$, $\|y'' - y'\| < \delta'$ and $\|v\| \leqslant 1$.

PROOF. By (2.1),

$$\begin{aligned}
&[\Delta_k h(y', v)]t - [\Delta_k h(y'', v)]t\\
&\qquad = \sum_{j=0}^{k}\left[1\Big/\left(\prod_{\substack{i\in[0,k]\\ i\neq j}}(t_j - t_i)\right)\right]\cdot[f(y' + t_j v) - f(y'' + t_j v)].
\end{aligned}$$

Since f is uniformly continuous in $N_{\delta/2}$ by Theorem 3.2(iii), we have the result immediately.

THEOREM 3.4. *Let A be any open subset of a Banach space E. If $f: A \to \mathbf{R}$ is n-convex, $n \geqslant 2$, and bounded in a neighbourhood N_δ of y_0 then for every $\varepsilon > 0$, there is a number $\delta' > 0$ such that*

$$\left|(\hat{f}^{(k)}y')v - (\hat{f}^{(k)}y'')v\right| < \varepsilon$$

whenever $k = 0, 1, \ldots, n-2$, $y', y'' \in N_{\delta/4}$, $\|y' - y''\| < \delta'$ and $\|v\| \leqslant 1$.

PROOF. By strong uniform continuity of $\Delta_k h(y, v)$, the limit (2.2) is uniform (cf. Theorem 3.2(i)) in the sense that there is a number $\delta'' > 0$ such that

$$\left|(\hat{f}^{(k)}y)v - (k!)\left[\Delta_k h(y, v)\right]t\right| < \varepsilon/3$$

whenever $y \in N_{\delta/4}$, $\|v\| \leqslant 1$ and $t = (t_0, \ldots, t_k)$ is a sequence of distinct numbers such that $0 < t_i \leqslant \delta''$. Then $(\hat{f}^{(k)}y')v$ and $(\hat{f}^{(k)}y'')v$ can be approximated uniformly by $(k!)[\Delta_k h(y', v)]t$ and $(k!)[\Delta_k h(y'', v)]t$, respectively using a fixed sequence of t's. The result now follows from Theorem 3.3.

THEOREM 3.5. *If A is an open subset of a Banach space E, $f: A \to \mathbf{R}$ is n-convex, $n \geqslant 3$, and f is bounded in a neighbourhood N_δ of y_0. Then given $\varepsilon > 0$ there is a $\bar{\delta} > 0$ such that*

$$\left|\left[\Delta_1 h(y', v)\right](0, t) - \left[\Delta_1 h(y'', v)\right](0, t)\right| < \varepsilon$$

whenever $\|y' - y_0\| < \delta/4$, $\|y'' - y_0\| < \delta/4$, $\|y' - y''\| < \bar{\delta}$, t a real number with $0 < t < \bar{\delta}$ and $\|v\| \leqslant 1$.

PROOF. For any real number $s > t$ we have the identity

$$\begin{aligned}\left[\Delta_1 h(y', v)\right](0, t) &- \left[\Delta_1 h(y'', v)\right](0, t)\\ &= \left[\Delta_1 h(y', v)\right](0, s) - \left[\Delta_1 h(y'', v)\right](0, s)\\ &\quad - (s - t)\left[\Delta_2 h(y', v)\right](0, t, s) + (s - t)\left[\Delta_2 h(y'', v)\right](0, t, s). \qquad (3.3)\end{aligned}$$

We may choose s first small enough so that the values of $\Delta_2 h(y', v)$ and $\Delta_2 h(y'', v)$ are uniformly bounded by Theorem 3.1, then choose s still smaller so that the absolute value of the $\Delta_2 h$ terms on the right side of (3.3) are less than or equal to $\varepsilon/4$, provided $0 < t < s$, $\|y' - y''\| < \delta/4$, $\|y' - y_0\| < \delta/4$ and $\|v\| \leqslant 1$. With a fixed choice of s, we may make the difference $y' - y''$ sufficiently small to make the $\Delta_1 h$ terms on the right side of (3.3) less than $\varepsilon/4$. Using Theorem 3.2, the number $\bar{\delta}$ is chosen to insure that $\|y' - y''\|$ is small enough and that $t < s$. Q.E.D.

4. Applications. The results of §3 give that an n-convex locally bounded function has a k-discrete bounded differential, $1 \leqslant k \leqslant n - 2$, at every point of the domain and that a one-sided $(n-1)$-discrete differential also exists and is bounded. This leads to the fact (proved in [4], and a detailed proof will be given elsewhere) that an n-convex locally bounded function admits a strong $(n-2)$-Taylor series expansion (for definition, see [5]). With the aid of a local representation theorem (see Theorem 3.2 in [5]) it follows that the kth Fréchet differential, $1 \leqslant k \leqslant n-2$, of an n-convex locally bounded function exists at every point of the domain. Later, this fact together with the assumption of a strong n-Taylor series expansion of f at a point leads to $(n-1)$th order Fréchet differentiability at that point. These results are proved by the author in [4] and the detailed proof will appear elsewhere.

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References

1. J. M. Ash, *A characterisation of the Peano derivatives*, Trans. Amer. Math. Soc. **149** (1970), 489–501.

2. P. L. Butzer and W. Kozakiewez, *On Riemann derivatives of integrable functions*, Canad. J. Math. **6** (1954), 572–581.

3. Z. Ciesielski, *Some properties of convex functions of higher order*, Ann. Polon. Math. **7** (1959), 572–581.

4. S. Dayal, *On local representations of functions on normed linear spaces*, Ph.D. thesis, Case Western Reserve University, Cleveland, Ohio, 1972.

5. ______, *A converse of Taylor's theorem for functions on Banach spaces*, Proc. Amer. Math. Soc. **65** (1977), 265–273.

6. A. Denjoy, *Sur l'intégration des coefficients différentiels d'ordre supérieur*, Fund. Math. **25** (1935), 273–362.

7. J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York, 1960.

8. N. Dunford and J. T. Schwartz, *Linear operators*. I, Interscience, New York, 1958.

9. R. Ger, *Convex functions of higher orders in Euclidean spaces*, Ann. Polon. Math. **25** (1972), 293–302.

10. ______, *n-convex functions in linear spaces*, Aequationes Math. **10** (1974), 172–176.

11. M. Z. Nashed, *On the representation and differentiability of operators*, Proceedings Conf. Constructive Theory of Functions (Budapest, 1969), Akad. Kiadó, Budapest, 1972, pp. 325–330.

12. ______, *Differentiability and relative properties of nonlinear operators: Some aspects of the role of differentials in nonlinear analysis*, Nonlinear Functional Anal. and Appl. (Proc. Advanced Sem., Math. Res. Center, Univ. of Wisconsin, Madison, 1970), Academic Press, New York, 1971, pp. 103–309.

13. C. E. Weil, *On approximation and Peano derivatives*, Proc. Amer. Math. Soc. **20** (1969), 487–490.

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