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BOUNDARY BEHAVIOR OF UNIVALENT FUNCTIONS SATISFYING A HÖLDER CONDITION

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ABSTRACT. Let f be univalent in the unit disk U and continuous in $U \cup T$, where $T = \partial U$. We prove that if f satisfies a Hölder condition, then each point in $f(T)$ is the image of at most finitely many points on T. The bound for the number of preimages depends in a sharp way on the Hölder exponent.

Let U be the unit disk and $T = \partial U$ be the unit circle. G. Piranian has asked the following question: Assume that f is analytic and univalent in U and continuous in $U \cup T$. Furthermore, assume that for some positive ε and some constant C, we have

$$|f(z_1) - f(z_2)| \leqslant C|z_1 - z_2|^{\varepsilon + 2/3}, \qquad z_1, z_2 \in U \cup T. \tag{1}$$

Is it true that every point in $f(T)$ is the image of at most two points on T?

The answer to Piranian's question is in fact yes and is a consequence of the case $p = 3$ in the following more general result.

THEOREM. *Let $p \geqslant 3$ be an integer and suppose that a is given and that $a > 2/p$. Let f be as above except that* (1) *is replaced by*

$$\limsup_{r \to 1-} (1 - r)^{-a}|f(e^{i\theta}) - f(re^{i\theta})| < \infty, \qquad |\theta| \leqslant \pi. \tag{2}$$

Then every point in $f(T)$ is the image of at most $p - 1$ points on T.

PROOF. Let us assume that $0 \in f(T)$ is the image of p different points $\{z_k\}_1^p$ on T. Then there exist p disjoint regions $\{\Omega_k\}_1^p$, all with 0 as a boundary point, such that $f^{-1}(\Omega_k)$ is a neighborhood in U of z_k, $k = 1, 2, \ldots, p$. We can also assume that the boundary of Ω_k is contained in $f(T) \cup \{w: |w| = R\}$, $k = 1, 2, \ldots, p$, where R is a small positive number.

Let u be the harmonic measure of the circle $\{w: |w| = R\}$ with respect to the set $f(U) \cap \{w: |w| < R\}$, i.e., the function which is harmonic in the set, 1 on $f(U) \cap \{w: |w| = R\}$ and 0 on $f(T) \cap \{w: |w| < R\}$ except possibly on a set of capacity zero. For $0 < r < R$, we define $\sigma_k(r) = \sup u(re^{i\theta})$, $re^{i\theta} \in \Omega_k$. We now use a deep result of M. Heins (cf. (4.1) in [1, p. 111]) which implies that there exists an absolute constant C such that

$$\prod_1^p \sigma_k(r) \leqslant C^p (r/R)^{p^2/2}, \qquad 0 < r < R.$$

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It follows that $\min_k \sigma_k(r) \leqslant C(r/R)^{p/2}$, $0 < r < R$. We conclude that in at least one of the regions $\{\Omega_k\}_1^p$, say Ω, there exists a sequence $\{r_n\}_1^\infty$ decreasing to zero such that

$$\max_{re^{i\theta} \in \Omega} u(re^{i\theta}) \leqslant C(r/R)^{p/2}, \qquad r \in \{r_n\}_1^\infty. \tag{3}$$

Let $z_0 \in T$ be such that $f(z_0) = 0$ and $f^{-1}(\Omega)$ is a neighborhood in U of z_0. We shall study the harmonic function $v(z) = u(f(z))$ near z_0. It is harmonic and positive in $f^{-1}(\Omega)$ and vanishes on T near z_0. It is well known that there exists a positive constant c such that

$$v(z) \geqslant c(1 - |z|), \qquad z \text{ near } z_0 \tag{4}$$

(cf. Heins [2, (9.3)] or Protter and Weinberger [3, Theorem 7, p. 65]).

We define $\gamma_n = \{w: |w| = r_n\} \cap \Omega$ and $\Gamma_n = f^{-1}(\gamma_n)$. Γ_n and parts of T near z_0 form the boundary of a region D_n: $\{D_n\}_1^\infty$ is a decreasing sequence of sets. For each n, we choose $t_n < 1$ such that $t_n z_0 \in \Gamma_n$. For $z \in \{t_n z_0\}$ and for large indices n, we have

$$\begin{aligned} c(1 - |z|) \leqslant v(z) = u(f(z)) &\leqslant CR^{-p/2}|f(z) - f(z_0)|^{p/2} \\ &\leqslant C(R, z_0)(1 - |z|)^{ap/2}. \end{aligned}$$

In this chain of inequalities, we first used (4), then (3), and finally (2).

Thus $(1 - t_n)^{1 - ap/2}$ is bounded from above as $n \to \infty$. Since $t_n \to 1$ as $n \to \infty$, this is possible only if $ap \leqslant 2$. Our assumption that $ap > 2$ now gives us a contradiction which proves the theorem.

EXAMPLE. Let $p \geqslant 3$ be given and consider the function $f(z) = z^{-1}((1 - z^p)^2/2)^{1/p}$ which defines a univalent mapping of U onto the complement in $\underline{C} \cup \{\infty\}$ of the set $\{w \in \underline{C}: 0 \leqslant |w| \leqslant 1, \arg w = (1 + 2k)\pi/p, k = 0, 1, \ldots, p - 1\}$ (the pth root is defined as real when $z \in U$ is real). For this function, the origin in the w-plane is the image of p points on T. If $z_0 \in T$ is such that $z_0^p = 1$, we have

$$|f(rz_0) - f(z_0)| = r^{-1}((1 - r^p)^2/2)^{1/p} \approx C_p(1 - r)^{2/p}, \qquad r \to 1 - .$$

For other points on T, there are better estimates. Thus an estimate of type (2) holds for f with $a = 2/p$.

This function f has a pole at the origin and is not analytic in U. However, it is now easy to construct a function analytic in U which satisfies the estimates above. We omit the details.

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