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RIEMANN R_1-SUMMABILITY OF INDEPENDENT, IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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ABSTRACT. Let $X, X_1, X_2, \ldots$ be i.i.d. random variables. It is shown that $E|X|\log^+\log^+|X| < \infty$ is a sufficient condition for Riemann R_1-summability of $\{X_n\}$ to EX. Counterexamples are provided which indicate that the strongest possible necessary condition of moment type is $E|X| < \infty$. However under weak regularity conditions on the tails of the distribution of X the sufficient condition is also shown to be necessary.

1. Introduction. Many regular forms of summability have been studied in great detail for independent, identically distributed random variables $X, X_1, X_2, \ldots$; the classic example being the strong law of large numbers which asserts that $E|X| < \infty$ iff $\{X_n\}$ is Cesàro $(C, 1)$ summable to EX. Lai [5] has shown that the existence of the first absolute moment is also necessary and sufficient for Cesàro (C, α) summability for all $\alpha \geqslant 1$ as well as for Abel summability, whereas Chow [1] in studying Euler (E, q), $q > 0$, summability and Borel summability found that the existence of a second moment was equivalent to summability. However, results for Riemann R_1-summability are not so tidy. Recall that a sequence of real numbers $\{A_n\}$ is R_1-summable to A if

$$\lim_{t\to 0^+} \frac{2}{\pi} \sum_{n=1}^{\infty} n^{-1}A_n \sin nt = A. \tag{1}$$

It is well known [3] that this is not a regular summability method. More importantly from our point of view, it is a discrete to continuous method in that the limit in (1) is taken through a continuum of values. In a recent paper Cuzick and Lai [2] have shown that a sequence of i.i.d. random variables $X, X_1, X_2, \ldots$ is R_1-summable to EX if $E|X|\log^+|X| < \infty$ and also that $E|X| < \infty$ is a necessary condition for summability. The approach there, as also followed here, is to treat (1) as a random Fourier series and, when $EX = 0$, to establish uniform convergence. The purpose of the present note is threefold:

(i) to show that

$$E|X|\log^+\log^+|X| < \infty \tag{2}$$

is a sufficient condition for R_1-summability,

(ii) to give examples which demonstrate that in general the condition $E|X| < \infty$ is the strongest possible necessary condition of moment type, i.e., given a monotone

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function $g(x)$ such that $\lim_{x\to\infty} g(x)/x = +\infty$ we can find an R_1-summable random variable X with $Eg(|X|) = +\infty$, and

(iii) to show that under a weak regularity condition on the smoothness of the tails of the distribution function of X the condition (2) is in fact necessary and sufficient.

2. Sufficient conditions. As the constant sequence is R_1-summable we may take $EX = 0$. Then R_1-summability follows from the continuity of

$$S(t) = \sum_{n=1}^{\infty} n^{-1}X_n \sin nt \tag{3}$$

which in turn follows from the uniform convergence of the series in (3).

THEOREM 1. *Assume* $EX = 0$ *and* $E|X|\log^+\log^+ < \infty$, *where* $\log^+ x = \max(0, \log x)$. *Then the series in* (3) *is uniformly convergent and* $\{X_n\}$ *is* R_1*-summable.*

PROOF. The proof is a refinement of the techniques used in [2]. Let X have distribution function F and for $\beta > 1$ set

$$X_n' = X_n I_{|X_n| < n/\log^\beta n}, \quad X_n'' = X_n I_{n/\log^\beta n < |X_n| < n},$$
$$X_n''' = X_n I_{|X_n| > n}$$

and write

$$\begin{aligned} S(t) &= \sum_{n=1}^{\infty} (X_n' - EX_n')n^{-1} \sin nt + \sum_{n=1}^{\infty} EX_n' n^{-1} \sin nt \\ &\quad + \sum_{n=1}^{\infty} X_n'' n^{-1} \sin nt + \sum_{n=1}^{\infty} X_n''' n^{-1} \sin nt \\ &= S_1(t) + S_2(t) + S_3(t) + S_4(t), \quad \text{say.} \end{aligned}$$

As $E|X| < \infty$ it follows from the Borel-Cantelli lemma that S_4 has only a finite number of nonzero terms and hence is uniformly convergent. Uniform convergence of S_3 follows from the fact that

$$\begin{aligned} E\sum_{n=3}^{\infty} n^{-1}|X_n''| &= \sum_{n=3}^{\infty} n^{-1} \int_{n/\log^\beta n < |x| < n} |x|\, dF(x) \\ &\leq \text{Const.} \int_{-\infty}^{\infty} \Bigg(\sum_{|x| < n < |x|\log^\beta |x|} n^{-1} \Bigg) |x|\, dF(x) \\ &\leq \text{Const.} \int_{-\infty}^{\infty} |x| \log^+\log^+|x|\, dF(x) < \infty. \end{aligned}$$

The uniform convergence of $S_1(t)$ follows from a result of Marcus [7] (see also [2, remarks after Theorem 3]) by checking that

$$\sum_{n=3}^{\infty} \left[\left\{ \sum_{k=n}^{\infty} E(X_k')^2 k^{-2} \right\}^{1/2} \Big/ n \log^{1/2} n \right] < \infty$$

which follows from the estimate

$$\begin{aligned}\sum_{k=n}^{\infty} E(X_n')^2 k^2 &= \sum_{k=n}^{\infty} k^{-2} \int_{|x|<k/\log^\beta k} x^2\, dF(x)\\ &= \int_{|x|<n/\log^\beta n} \left(\sum_{k=n}^{\infty} k^{-2}\right) x^2\, dF(x)\\ &\quad + \int_{|x|>n/\log^\beta n} \left(\sum_{k/\log^\beta k>|x|} k^{-2}\right) x^2\, dF(x)\\ &= O(\log n)^{-\beta} \quad \text{since } \beta > 1.\end{aligned}$$

As $EX = 0$, we may replace EX_n' by $EXI_{|X|>n/\log^\beta n}$ in S_2. Interchanging summation and expectation as before we find the Nth partial sum of $S_2(t)$ equals

$$\int_{-\infty}^{\infty} \left(\sum_{n/\log^\beta n<x}^{N} n^{-1} \sin nt\right) x\, dF(x). \tag{4}$$

If we compute

$$C(i, t) = \sup_j \left| \sum_{i<n<j} n^{-1} \sin nt \right|$$

and

$$C = \sup_{i,t} C(i, t),$$

it is easy to check that $C < \infty$ and $C(i, t)\downarrow 0$ as $i\uparrow\infty$ uniformly for t on compact sets not containing $\{2\pi n\}_{n=-\infty}^{\infty}$. From this it follows that $S_2(t)$ is uniformly convergent on such sets and we need only concern ourselves with uniform convergence in a neighborhood of the origin, i.e. given $\varepsilon > 0$, it is enough to find a $\delta > 0$ such that when $|t| < \delta$, (4) is less than ε for all N. Choose D so that

$$\int_{|x|>D} |x|\, dF(x) < \varepsilon/2C.$$

Then the integral (4) on the set $|x| > D$ is less than $\varepsilon/2$ and it suffices to show that for all N

$$\int_{|x|<D} \left(\sum_{N<n/\log^\beta n<x} n^{-1} \sin nt\right) x\, dF(x) < \varepsilon/2$$

for all t sufficiently small. This follows immediately from the convergence to zero of the integrand as $t \to 0$ uniformly for $|x| < D$, as the series contains only a finite number of terms for $|x|$ bounded.

3. Necessary conditions. Cuzick and Lai [2] noted that $E|X| < \infty$ was necessary for R_1-convergence. The following example indicates that no stronger necessary condition of moment type can be found. Let $g(x)$ be any function such that $g(x)/x\uparrow\infty$ as $x \to \infty$. Choose any $0 < \varepsilon < 1$ and let n_k be the first $n > 2n_{k-1}$ and of the form $2^j n$ for some $j > 1$ such that $g(n_k)/n_k > k^{2+\varepsilon}$. Let X have mass $M/(n_k k^{2+\varepsilon})$ at $\pm n_k$ with M chosen so that the total mass is unity. Then $E|X| < \infty$ but $Eg(|X|) = +\infty$. As $E|X_n| < \infty$, by the Borel-Cantelli lemma we may replace

X_n by $X_n' = X_n I_{|X_n|<n}$ and change only a finite number of terms. Now for $n_k \leq 2^n < n_{k+1}$ compute

$$s_n = \left(\sum_{j=2^n+1}^{2^{n+1}} E(X_j'/j)^2 \right)^{1/2} = O\left(\frac{n_k}{2^n k^{2+\varepsilon}} \right)^{1/2}.$$

Then we may choose $s_n^* > s_n$, with s_n^* nonincreasing so that

$$\sum_{n=1}^{\infty} s_n^* = \sum_{k=1}^{\infty} \left(\sum_{n_k \leq 2^n < n_{k+1}} s_n^* \right) = O\left(\sum_{k=1}^{\infty} k^{-(1+\varepsilon/2)} \right) < \infty.$$

It follows from a result of Kahane [4, p. 65, Remark 2] that

$$\sum_{n=1}^{\infty} X_n' n^{-1} \sin nt$$

is uniformly convergent and thus $\{X_n\}$ is R_1-summable.

This result relies heavily on the lacunary behavior of the distribution of X. If some mild conditions are placed on the regularity of the tails of the distribution of X, it is possible to show that (2) is also necessary for R_1-summability.

THEOREM 2. *Assume that for all n sufficiently large*

$$\int_{2^n<|x|\leq 2^{n+1}} |x| \, dF(x) \text{ is nonincreasing as } n \uparrow \infty. \tag{5}$$

Then $\{X_n\}$ *is Riemann* R_1*-summable* $\Leftrightarrow E|X|\log^+\log^+|X| < \infty$.

PROOF. We need only establish that under (5), condition (2) is necessary for summability. First assume X is symmetric. We shall need the results of §3 in [2]. Set $W_n = XI_{0\leq X\leq n}$ and let G_n be the unique solution of the equation

$$nE\left(\min\left\{|W_n/G_n|, (W_n/G_n)^2\right\}\right) = 1. \tag{6}$$

If $\{X_n\}$ is summable it follows from Theorem 5 of [2] that

$$\sum_{n=1}^{\infty} \gamma_n < \infty \tag{7}$$

where $\gamma_n = G_{2^n}/2^n$. We aim to show that

$$\gamma_n = o\left(\frac{1}{n}\right) \tag{8}$$

which follows from (7) if we can establish that γ_n is "almost monotone" in the sense that there exists an n_0 and $K_1 > 0$ such that

$$\gamma_n \geq K_1 \gamma_m \quad \text{for all } m \geq n \geq n_0. \tag{9}$$

To prove this we need to verify the existence of an n_0 and $K_2 > 0$ such that for all $n \geq j \geq n_0$

$$2^{-n} \int_0^{2^n} x^2 \, dF(x) \geq K_2 2^{-(n+k)} \int_0^{2^{n+k}} x^2 \, dF(x), \quad \text{all } k \geq 0 \tag{10}$$

and

$$\int_{2^j}^{2^n} x \, dF(x) \geq K_2 \int_{2^{j+k}}^{2^{n+k}} x \, dF(x), \quad \text{all } k \geq 0. \tag{11}$$

Equation (11) follows immediately from (5) and (10) follows from (5) upon noting that

$$\begin{aligned}\int_{2^{n_0+1}}^{2^n} x^2\,dF(x) &= \sum_{m=n_0}^{n} \int_{2^{m-1}}^{2^m} x^2\,dF(x)\\ &\geqslant \sum_{m=n_0}^{n} 2^{m-1}\int_{2^{m-1}}^{2^m} x\,dF(x) \geqslant \sum_{m=n_0}^{n} 2^{m-1}\int_{2^{m+k-1}}^{2^{m+k}} x\,dF(x)\\ &\geqslant \sum_{m=n_0}^{n} 2^{k-2}\int_{2^{m+k-1}}^{2^{m+k}} x^2\,dF(x) \geqslant \frac{1}{4}2^{-k}\int_{2^{n_0+1}}^{2^{n+k}} x^2\,dF(x).\end{aligned}$$

Now for $0 < G \leqslant 2^n$, define

$$g_n(G) = \int_0^G \frac{x^2}{G}\,dF(x) + \int_G^{2^n} x\,dF(x). \tag{12}$$

Then g_n is nonincreasing in G. Choose β_n of the form 2^i for some integer i such that $\frac{1}{2}\beta_n < G_{2^n} \leqslant \beta_n$. Then using (6)

$$\gamma_n = g_n(G_{2^n}) \geqslant g_n(\beta_n) \tag{13}$$

which from (10) and (11) is greater than or equal to $K_2 g_{n+k}(2^k\beta_n)$ for all $k \geqslant 0$. If $2^k\beta_n \leqslant G_{2^{n+k}}$ then

$$K_2 g_{n+k}(2^k\beta_n) \geqslant K_2 g_{n+k}(G_{2^{n+k}}) = K_2\gamma_{n+k}$$

so that (9) holds. Of course (9) also holds when $2^k\beta_n > G_{2^{n+k}}$ since then $G_{2^n} > \frac{1}{2}\beta_n > 2^{-(k+1)}G_{2^{n+k}}$ so that $\gamma_n = G_{2^n}/2^n \geqslant \frac{1}{2}\gamma_{n+k}$.

Now define $G(x) = G_{2^n}$ for $x = n$ and by linear interpolation for nonintegral x. It is easily checked that G is nondecreasing. From (8) it follows that there exists $K_3 > 0$, such that for sufficiently large x

$$G^{-1}(x) \geqslant \log x + K_3 \log\log x. \tag{14}$$

It follows from (7), (13) and (12) that

$$\sum_{n=1}^{\infty} \int_{G(n)}^{2^n} x\,dF(x) < \infty$$

and a Fubini argument shows

$$\int^{\infty} \left[G^{-1}(x) - \log x\right] x\,dF(x) < \infty$$

from whence (2) follows with the help of (14). This establishes the theorem when X is symmetric. In general let m be a median of X and for $2^k \leqslant n < 2^{k+1}$ define

$$Y_n = (X_n - m)I_{|X_n - m| < 2^{k-1}}$$

and let Z_n be the symmetrized version of Y_n, i.e. $Z_n = Y_n - Y_n'$ where Y_n, Y_n' are i.i.d. Then $\{X_n\}$-summable $\Rightarrow$ $\{Z_n\}$-summable

$$\Rightarrow \sum_{k=1}^{\infty} \gamma_k^Z < \infty \quad \text{as at (7)} \tag{15}$$

where $\gamma_k^Z = G_{2^k}^Z/2^k$ and $G_{2^k}^Z$ is the unique solution of (6) with W_n replaced by Z_{2^k}.

Define γ_k^Y and $G_{2^k}^Y$ similarly. Then integration by parts and the weak symmetrization inequality [6, p. 245] gives

$$\begin{aligned}
\gamma_k^Z &= E\left(\min\left\{Z_{2^k}^2/G_{2^k}, |Z_{2^k}|\right\}\right) = \int_0^\infty \min\left\{x^2/G_{2^k}^Z, x\right\} dF_{|Z_{2^k}|}(x) \\
&\geqslant \frac{1}{2}\int_0^\infty \left(1 - F_{|Z_{2^k}|}(x)\right)\min\left\{2x/G_{2^k}^Z, 2\right\} dx \\
&\geqslant \frac{1}{4}\int_0^\infty \left(1 - F_{|Y_{2^k}|}\left(\frac{x}{2}\right)\right)\min\left\{2x/G_{2^k}^Z, 2\right\} dx \\
&\geqslant \frac{1}{16}\int_0^\infty \left(1 - F_{|Y_{2^k}|}(x)\right)\min\left\{2x/G_{2^k}^Z, 2\right\} dx \\
&\geqslant \frac{1}{16} E\left(\min\left\{Y_{2^k}^2/G_{2^k}^Z, |Y_{2^k}|\right\}\right).
\end{aligned}$$

From this it can be shown as in the argument following (13) that

$$\gamma_k^Z \geqslant \tfrac{1}{8}\gamma_k^Y. \tag{16}$$

It is easily verified from (6) that for k large there exists $K_3 > 0$ such that

$$G_{2^k}^Y > K_3 2^{k/2}. \tag{17}$$

Now for $G_{2^k}^Y > 2m$ we have

$$\begin{aligned}
\gamma_k^Y &= E\left(\min\left\{Y_{2^k}^2/G_{2^k}^Y, |Y_n|\right\}\right) \\
&= E\left(\min\left\{(X-m)^2 I_{|X-m|<2^{k-1}}/G_{2^k}^Y, |X-m| I_{|X-m|<2^{k-1}}\right\}\right) \\
&\geqslant \tfrac{1}{2} E\left(\min\left\{X^2 I_{|X|<2^k}/G_{2^k}^Y, |X| I_{|X|<2^k}\right\} - m^2/G_{2^k}^Y\right) \\
&\geqslant \tfrac{1}{2}\left(\gamma_k^X - m^2/G_{2^k}^Y\right).
\end{aligned}$$

Combining (15), (16) and (17) gives

$$\sum_{k=1}^\infty \gamma_k^X < \infty. \tag{18}$$

If we let $|V| \overset{\mathcal{D}}{=} |X|$ and let V be symmetric, i.e. $V = \pm X$, then $\gamma_k^V = \gamma_k^X$ and, as V satisfies (5) and is symmetric, we may use the proof previously given to see that (18) implies $\{V_n\}$ is summable

$$\Rightarrow E|X|\log^+\log^+|X| = E|V|\log^+\log^+|V| < \infty$$

which completes the proof in general.

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