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MARTINGALE CONVERGENCE VIA THE SQUARE FUNCTION

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ABSTRACT. By exploiting the natural setting of a convergence theorem of Burkholder, a direct and elementary proof of the theorem is given. This proof is also new for the martingale convergence theorem and the martingale transform convergence theorem which are corollaries to the above-mentioned convergence theorem.

1. Introduction. An important generalization of Doob's martingale convergence theorem is Burkholder's **[4]** martingale transform convergence theorem. In **[4]**, Burkholder also proves the following result:

THEOREM 1. *Let* $(\Omega, \mathfrak{F}, P)$ *be a probability space and* $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots$ *a nondecreasing sequence of sub-*σ*-algebras of* $\mathfrak{F}$. *Let* $f = (f_1, f_2, \ldots)$ *and* $g = (g_1, g_2, \ldots)$ *be two martingales relative to* $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots$. *Suppose* $\|f\|_1 < \infty$ *and* $S_n(g) \leqslant S_n(f)$ *for* $n = 1, 2, \ldots$. *Then* g *converges a.s.*

Here and throughout the rest of this paper, we adopt the following notation: for a martingale $f = (f_1, f_2, \ldots)$ relative to $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots$ and with difference sequence $d = (d_1, d_2, \ldots)$, $\|f\|_p = \sup_n(E|f_n|^p)^{1/p}$, $S_n(f) = (\sum_{i=1}^n d_i^2)^{1/2}$, $S(f) = \sup_n S_n(f)$, $f^* = \sup_n|f_n|$, $d^* = \sup_n|d_n|$, $f^\tau = (f_1^\tau, f_2^\tau, \ldots)$ denotes f stopped at τ, $d^\tau = (d_1^\tau, d_2^\tau, \ldots)$ denotes the difference of f^τ, and E_n denotes the conditional expectation operator given $\mathfrak{F}_n$ for $n \geqslant 1$ and $E_0 = E$.

It is easy to see that Theorem 1 is a generalization of both the martingale convergence theorem and the martingale transform convergence theorem.

Different proofs of the martingale convergence theorem abound in the literature (see, for example, **[2]**, **[9]**, **[10]** and **[11]**). There are also several proofs of the martingale transform convergence theorem some of which are extensions of the proofs of the martingale convergence theorem (see, for example, **[3]**, **[4]**, **[8**, pp. 72–74] and **[10]**). However there does not seem to exist in the literature other proofs than Burkholder's original proof of Theorem 1. The latter uses the martingale transform convergence theorem. (It is possible to deduce Theorem 1 from the deep inequality $\|f^*\|_1 \leqslant C\|S(f)\|_1$ of Davis **[7]** by using a stopping time and then applying the martingale convergence theorem. But this is not our point.)

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Actually the setting of Theorem 1 is very natural. It enables one to link the convergence of a martingale to its square function and use the elementary fact that the square function of an L_1-bounded martingale is finite a.s. (see, for example, **[1]**, **[5**, p. 21], and **[6]**; see also the remark at the end of this paper). Thus the purpose of this paper is to give a direct an elementary proof of Theorem 1 which is also a new proof of the martingale convergence theorem and the martingale transform convergence theorem.

2. Proof. Let $d = (d_1, d_2, \ldots)$ and $e = (e_1, e_2, \ldots)$ be the difference sequences of f and g respectively. Define $\tau = \inf\{n\colon |f_n| > \lambda \text{ or } S_n(f) > \lambda\}$ where $\lambda > 0$. Then

$$\begin{aligned} |e_n^\tau| &\leqslant S_n(g^\tau) = S_{\tau\wedge n}(g) \leqslant S_{\tau\wedge n}(f) \\ &= S_n(f)I(\tau > n) + S_\tau(f)I(\tau \leqslant n) \\ &\leqslant \lambda + |d_\tau| I(\tau \leqslant n) \end{aligned}$$

where $I(A)$ denotes the indicator function of the set A. Therefore

$$e^{\tau *} \leqslant \lambda + |d_\tau| I(\tau < \infty) \leqslant 2\lambda + |f_\tau| I(\tau < \infty)$$

and so

$$\begin{aligned} Ee^{\tau *} &\leqslant 2\lambda + E|f_\tau| I(\tau < \infty) \\ &= 2\lambda + \sup_n E|f_{\tau\wedge n}| I(\tau < \infty) \\ &\leqslant 2\lambda + \sup_n E|f_{\tau\wedge n}| \leqslant 2\lambda + \|f\|_1 < \infty \end{aligned}$$

where the inequality $\sup_n E|f_{\tau\wedge n}| \leqslant \|f\|_1$ follows from Doob's optional sampling theorem.

Now it is not difficult to see that

$$E \sum_{n=1}^{\infty} \frac{e_n^{\tau 2}}{U_n^2} \leqslant E \int_0^\infty \frac{dx}{(1+x)^2} = 1$$

where $U_n = 1 + S_n^2(g^\tau)$. So if we let

$$\xi_n = U_n^{-1} e_n^\tau - E_{n-1} U_n^{-1} e_n^\tau$$

and define the martingale $h = (h_1, h_2, \ldots)$ relative to $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots$ by $h_n = \sum_{i=1}^n \xi_i$, it will follow that h is L_2-bounded. Indeed,

$$\begin{aligned} \|h\|_2^2 &= \sup_n Eh_n^2 = \sum_{n=1}^\infty E\xi_n^2 \\ &= \sum_{n=1}^\infty EE_{n-1}\xi_n^2 \leqslant \sum_{n=1}^\infty EE_{n-1} U_n^{-2} e_n^{\tau 2} \\ &= E \sum_{n=1}^\infty U_n^{-2} e_n^{\tau 2} < \infty. \end{aligned}$$

Hence h converges a.s. To emphasize the elementary character of this proof, we note that the a.s. convergence of an L_2-bounded martingale is a trivial generalization of a classical theorem concerning the a.s. convergence of sums of independent random variables. Now

$$E\sum_{n=2}^{\infty}|E_{n-1}U_n^{-1}e_n^{\tau}| = E\sum_{n=2}^{\infty}|E_{n-1}U_n^{-1}e_n^{\tau} - E_{n-1}U_{n-1}^{-1}e_n^{\tau}|$$
$$\leqslant E\sum_{n=2}^{\infty}E_{n-1}e^{\tau*}\left(U_{n-1}^{-1} - U_n^{-1}\right) \leqslant Ee^{\tau*} < \infty$$

and so $\sum_{n=1}^{\infty}E_{n-1}U_n^{-1}e_n^{\tau}$ converges absolutely a.s. It follows that $\sum_{n=1}^{\infty}U_n^{-1}e_n^{\tau}$ converges a.s. and in particular a.s. on $\{\tau = \infty\} = \{S(f) \leqslant \lambda, f^* \leqslant \lambda\}$. But on $\{\tau = \infty\}$, $e_n^{\tau} = e_n$ and $S_n(g^{\tau}) = S_n(g)$. So we have $\sum_{n=1}^{\infty}[1 + S_n^2(g)]^{-1}e_n$ converges a.s. on $\{S(f) \leqslant \lambda, f^* \leqslant \lambda\}$ and hence on

$$\{S(f) < \infty, f^* < \infty\} = \bigcup_{\lambda>0}\{S(f) \leqslant \lambda, f^* \leqslant \lambda\}.$$

Now $\{S(g) < \infty\} \supset \{S(f) < \infty\}$ and it is easy to check that if $\sum_{n=1}^{\infty}a_n$ is a convergent series of real numbers and $\{b_n\}$ is a monotone sequence of real numbers with finite limit then $\sum_{n=1}^{\infty}a_nb_n$ converges. Using these we have g converges a.s. on $\{S(f) < \infty, f^* < \infty\}$. But by Doob's maximal inequality $\|f\|_1 < \infty$ implies $f^* < \infty$ a.s. and by Austin's result **[1]** $\|f\|_1 < \infty$ implies $S(f) < \infty$ a.s. (see also **[5**, p. 21] and **[6]**). It follows that g converges a.s. This proves the theorem.

REMARK. In **[1]**, although the martingale convergence theorem is quoted in the last step of the proof, it is Doob's maximal inequality that is really needed. Similarly, in **[5]**, it is not necessary to use Lemma 2.1 to prove Theorem 3.1; the inequality (2.2) is sufficient.

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