PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 1, September 1981

CHAINS AND DISCRETE SETS IN ZERO-DIMENSIONAL COMPACT SPACES

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ABSTRACT. Let X be a compact zero-dimensional space and let $B(X)$ denote the Boolean algebra of all clopen subsets of X. Let κ be an infinite cardinal. It is shown that if $B(X)$ contains a chain of cardinality κ then $X \times X$ contains a discrete subset of cardinality κ. This complements a recent result of J. Baumgartner and P. Komjath relating antichains in $B(X)$ to the π-weight of X.

1. Introduction. Let X be a compact zero-dimensional space, and let $B(X)$ denote the Boolean algebra of clopen subsets of X. In this note we are interested in the relations between certain algebraic aspects of the Boolean algebra $B(X)$ and the topological properties of X. Specifically, we are concerned with chains and antichains in $B(X)$ and their connection with cardinal invariants of the space X.

Our set-theoretic and topological terminology and notation are standard. The cardinality of a set S is denoted by $|S|$. All cardinal numbers considered in this paper are assumed to be infinite. Our basic reference for cardinality properties of topological spaces is **[J]**.

For the reader's convenience we now recall several notions which will be useful in the sequel.

Recall that a topological space D is said to be *discrete* if every point of D is open in D. Thus if D is a subspace of a space X, then D is discrete if for every point x of D there is an open set G_x in X such that $G_x \cap D = \{x\}$. If X is a topological space the *spread* of X, denoted by $s(X)$, is defined by

$$s(X) = \sup\{\kappa: X \text{ contains a discrete subset of cardinality } \kappa\}.$$

If X is a space then the *density character* of X, denoted by $d(X)$, is defined by $d(X) = \min\{\kappa: X \text{ has a dense subset of cardinality } \kappa\}$. If $d(X) = \omega$ we say that X is *separable*. The *hereditary density character* of X, denoted by $\bar{d}(X)$, is defined by $\bar{d}(X) = \sup\{d(S): S \subseteq X\}$. If $\bar{d}(X) = \omega$ we say that X is *hereditarily separable*.

A family Π of nonempty open subsets of a space X is called a *π-base* for X if every nonempty open subset of X contains a member of Π. The *π-weight* of X, denoted by $\pi(X)$, is defined by $\pi(X) = \min\{\kappa: X \text{ has a } \pi\text{-base of size } \kappa\}$. The *hereditary π-weight* of X, denoted by $\bar{\pi}(X)$, is defined to be

$$\bar{\pi}(X) = \sup\{\pi(S): S \subseteq X\}.$$

Received by the editors October 15, 1980 and, in revised form, January 21, 1981.
1980 *Mathematics Subject Classification*. Primary 54A25; Secondary 54O30.
Key words and phrases. Discrete set, compact zero-dimensional space, chain, antichain, strong antichain.

[1]The second author dedicates his research in this paper to Sharon and Gillian.

Clearly if D is a discrete space of cardinality κ then the density character of D is κ. Therefore $s(X) \leqslant \bar{d}(X)$. If Π is a π-base for X of cardinality κ and if for each $P \in \Pi$ we choose a point x_P in P, then the set $D = \{x_P: P \in \Pi\}$ is a dense subset of X of cardinality at most κ. This shows that $d(X) \leqslant \pi(X)$ and hence that $\bar{d}(X) \leqslant \bar{\pi}(X)$.

Let X be a space and let $(T, \leqslant)$ be a totally ordered set. Let $S = \{x_t: t \in T\}$ be a subset of X indexed by T. S is said to be *totally separated by* T if $\{x_s: s \leqslant t\}$ is open in S for every t in T. We say that S is a totally separated subspace of X.

If B is a Boolean algebra then elements x and y of B are said to be *comparable* if either $x \leqslant y$ or $y \leqslant x$; otherwise x and y are said to be *incomparable*. A subset S of B is called a *chain* if every two elements of S are comparable. A subset S of B is called an *antichain* if no two elements of S are comparable. Finally, a subset S of B is called a *strong antichain* if no element x of S is contained in the union of any finite subset of $S - \{x\}$; that is, $x \nleqslant \bigvee F$ for any finite subset F of $S - \{x\}$. Every strong antichain is an antichain but not conversely. The *height* of B and the *width* of B are defined respectively by $h(B) = \sup\{\kappa: B$ contains a chain of cardinality $\kappa\}$ and $w(B) = \sup\{\kappa: B$ contains an antichain of cardinality $\kappa\}$.

2. Chains in $B(X)$ and discrete subsets of $X \times X$. Throughout this section we assume that X is a zero-dimensional compact space.

In **[BK]** it is shown that if all antichains of $B(X)$ have cardinality at most κ then X has a π-base of cardinality at most κ. That is, $\pi(X) \leqslant w(B(X))$. In **[IN]** it is pointed out that this result actually holds for hereditary π-weight: $\bar{\pi}(X) \leqslant w(B(X))$. In particular, if all antichains of $B(X)$ are countable, then all subspaces of X have countable π-weight and hence X is hereditarily separable. A familiar example can be used to show that the above result cannot be improved to an equality: the Alexandroff-Urysohn "double-arrow" space (which is the same as the top and bottom of the lexicographically ordered unit square), is a compact zero-dimensional space which has an antichain of c clopen sets. Furthermore, this space is hereditarily separable and first countable, and hence its hereditary π-weight is ω. This example is also discussed in 9C of **[E]**.

We will now establish a relation between chains of $B(X)$ and cardinal invariants of the space X.

2.1 THEOREM. *Let κ be an infinite cardinal. If $B(X)$ contains a chain of cardinality κ then X contains a totally separated subspace of cardinality κ, and $X \times X$ contains a discrete subset of cardinality κ. In particular, $h(B(X)) \leqslant s(X \times X)$.*

PROOF. Let Λ be a chain in $B(X)$ with $|\Lambda| = \kappa$. By deleting $\varnothing$ if necessary we may assume that every member of Λ is nonempty. Let $A \in \Lambda$. We claim that $\bigcup\{B: B \in \Lambda \text{ and } B \subsetneq A\} \subsetneq A$. For A is compact, being closed in X, and so, if the preceding union where equal to A it follows that there would exist a finite number $B_1, B_2, \cdots, B_n$ of members of Λ with $B_i \subsetneq A$ for all $i = 1, 2, \ldots, n$ such that $B_1 \cup B_2 \cup \cdots \cup B_n = A$. But Λ is a chain under inclusion so there is a largest member among $B_1, B_2, \ldots, B_n$, say B_j. Thus $B_j = A$. But this is impossible, since B_j is a proper subset of A. This proves our claim. Thus for each A in Λ we may choose a point $x_A \in A - \bigcup\{B: B \in \Lambda \text{ and } B \subsetneq A\}$. We note that the subspace

$S = \{x_A: A \in \Lambda\}$ is totally separated by Λ under inclusion–in fact, for all A in Λ we have $\{x_B: B \in \Lambda \text{ and } B \subseteq A\} = A \cap S$, which implies that the set $\{x_B: B \in \Lambda \text{ and } B \subseteq A\}$ is open in S. Thus S is a totally separated subspace of X having cardinality κ. This establishes the first part of the theorem.

Now, we can repeat the above argument applied to the chain $\Lambda^* = \{X - A: A \in \Lambda\}$ (after deleting X from Λ if necessary to ensure that each member of Λ is proper). The result is a set of points $\{y_A: A \in \Lambda\}$ such that, for all $A, y_A \in X - A$, and $y_A \notin X - B$ for any B in Λ with $X - B \subsetneq X - A$. That is, $y_A \notin A$ and $y_A \in B$ for all B in Λ such that $A \subsetneq B$. We now claim that the set $D = \{(x_A, y_A): A \in \Lambda\}$ is a discrete subset of $X \times X$, proving that $X \times X$ has a discrete subset of cardinality κ as desired. This follows from the fact that $A \times (X - A)$ is open in $X \times X$ and $A \times (X - A) \cap D = \{(x_A, y_A)\}$: for, if $B \subsetneq A$ then $y_B \in A$ and so $(x_B, y_B) \notin A \times (X - A)$, while if $A \subsetneq B$ then $x_B \notin A$ and so again $(x_B, y_B) \notin A \times (X - A)$. This completes the proof.

An easy example shows that the inequality in our theorem cannot be improved to an equality: Let X be the one-point compactification of the discrete space of cardinality κ. Since all clopen subsets of X are either finite or cofinite, it follows that all chains in $B(X)$ are countable. However $X \times X$, and indeed X itself, contains a discrete subset of cardinality κ.

It is worth noting that the existence of a discrete subset of X of cardinality κ does not follow from the existence of a chain in $B(X)$ of cardinality κ. The double-arrow space serves as an example here also; it contains a chain of c clopen sets but contains no uncountable discrete subsets, since it is hereditarily separable.

As for discrete subsets of X we have the following simple result.

2.2 THEOREM. *Let X be a compact zero-dimensional space and let κ be an infinite cardinal. Then X contains a discrete subset of cardinality κ if and only if $B(X)$ contains a strong antichain of cardinality κ.*

PROOF. Suppose X contains a discrete subset D with $|D| = \kappa$. Then for each x in D there is a clopen set A_x in X such that $A_x \cap D = \{x\}$. Clearly the collection $\Lambda = \{A_x: x \in D\}$ is a strong antichain in $B(X)$, since, for any x in D and any finite subset F of $D - \{x\}$, we have $x \in A_x - \bigcup \{A_y: y \in F\}$ and so $A_x \nsubseteq \bigcup \{A_y: y \in F\}$. Thus Λ is a strong antichain of cardinality κ.

Conversely, suppose Λ is a strong antichain in $B(X)$ of cardinality κ. Let A be any member of Λ. The family of clopen sets $\{A - B: B \in \Lambda - \{A\}\}$ has the finite intersection property, since A is not contained in the union of any finite number of members of $\Lambda - \{A\}$. Therefore, by compactness, $\bigcap \{A - B: B \in \Lambda - \{A\}\} \neq \varnothing$. Choose a point x_A in that intersection. Doing this for each A in Λ we construct a subset $D = \{x_A: A \in \Lambda\}$ of X such that $A \cap D = \{x_A\}$ for all A in Λ. Thus the set D is a discrete set of cardinality κ.

Combining 2.1 and 2.2 and using the obvious fact that $X \times X$ contains a discrete set of cardinality κ whenever X does, we have the following corollary.

2.3 COROLLARY. *If $X \times X$ contains no discrete subsets of cardinality $> \kappa$ then all chains and all strong antichains of $B(X)$ have cardinality at most κ.*

REMARK. The conclusion of 2.2 has an obvious generalization to arbitrary compact spaces. One replaces the concept of a strong antichain of clopen sets by a set of pairs $\{(F_i, G_i): i \in I\}$ where F_i is closed and G_i is open and $F_i \subseteq G_i$ and such that for any i and any finite subset J of $I - \{i\}$, $F_i \not\subseteq \bigcup \{G_j: j \in J\}$. There does not seem to be any natural generalization of 2.1 to arbitrary compact spaces.

REFERENCES

[BK] J. Baumgartner and P. Komjath, *Boolean algebras in which every chain and every antichain is countable* (to appear).

[E] R. Engleking, *Outline of general topology*, North-Holland, Amsterdam, 1968.

[IN] M. Ismail and P. Nyikos, *Countable small rank and cardinal invariants* II (preprint).

[J] I. Juhasz, *Cardinal functions in topology*, Math. Centre Tracts, No. 34, Math. Centre, Amsterdam, 1971.

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