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INTERPOLATION SPACES AND UNITARY REPRESENTATIONS

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ABSTRACT. Let G be a Lie group, π a unitary representation of G on a Hilbert space $\mathcal{H}(\pi)$, and $\mathcal{H}^k(\pi)$ the subspace of C^k vectors for π. By quadratic interpolation there is a continuous scale $\mathcal{H}^s(\pi)$, $s > 0$, of G-invariant Hilbert spaces. When $G = H \cdot K$ is a semidirect product of closed subgroups, then it is proved that $\mathcal{H}^s(\pi) = \mathcal{H}^s(\pi|_H) \cap \mathcal{H}^s(\pi|_K)$ for $s > 0$. For solvable G this gives a characterisation of $\mathcal{H}^s(\pi)$ in terms of smoothness along one-parameter subgroups, and an elliptic regularity result.

1. Introduction. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $U(\mathfrak{g})$ the complexified universal enveloping algebra of $\mathfrak{g}$. If π is a strongly continuous representation of G on a Banach space $\mathcal{H}(\pi)$, one has for each positive integer k the subspace $\mathcal{H}^k = \mathcal{H}^k(\pi)$ of C^k vectors for π, and the space $\mathcal{H}^\infty(\pi) = \bigcap \mathcal{H}^k$ of C^∞ vectors. These spaces are G-invariant. Furthermore, $\mathcal{H}^k$ can be normed as a Banach space so that $g \mapsto \pi(g)|_{\mathcal{H}^k}$ is strongly continuous (cf. [3]).

It is natural to consider interpolation spaces between $\mathcal{H}^k$ and $\mathcal{H}^{k+1}$, constructed either by the "complex" or "real" method. For general Banach space representations, this gives a profusion of different spaces (cf. [2], [10] for the much-studied case $G = \mathbf{R}^n$). For unitary representations one may use the "quadratic interpolation functor" [9] to obtain a continuous interpolating scale of Hilbert spaces $\mathcal{H}^s$, $s > 0$ (cf. [4]), which are G-invariant. (For noncommutative G, however, the G action on $\mathcal{H}^s$ is nonunitary for $s > 0$, when the adjoint representation is nonunitary.) The main result of this note is the following

REDUCTION THEOREM. *If $G = H \cdot K$ is a semidirect product of closed subgroups H and K, then for any unitary representation π and $s > 0$,*

$$\mathcal{H}^s(\pi) = \mathcal{H}^s(\pi|_H) \cap \mathcal{H}^s(\pi|_K).$$

We shall prove this theorem in §5, using basic properties of diffusion semigroups generated by Laplace operators, which are recalled in §3, and a "noncommutative interpolation" theorem of P. Grisvard [7], which is stated in the form we need it in §4. This approach seems quite natural, in view of the fact that for $s > 0$, $\mathcal{H}^s(\pi)$ is the domain of the operator $A^{s/2}$, where A is the closure of $\pi_\infty(-\Delta)$, Δ a Laplace operator on G (cf. §5 for notation).

2. Applications. We turn now to some consequences of the Reduction Theorem, when G is solvable. First we describe $\mathcal{H}^s(\pi)$, $0 < s < 1$, in the case $G = \mathbf{R}$. Here the result is known (cf. [10, Proposition 4, p. 139]).

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PROPOSITION 2.1. *If π is a unitary representation of* $\mathbf{R}$ *and* $0 < s < 1$, *then* $u \in \mathcal{H}^s(\pi)$ *if and only if*

$$\int_{-\infty}^{\infty} \|\pi(x)u - u\|^2 |x|^{-2s-1}\, dx < \infty. \tag{2.1}$$

PROOF. By Bochner's theorem, there is a unique positive finite measure μ on $\mathbf{R}$ such that

$$(\pi(x)u, u) = \int_{-\infty}^{\infty} e^{ix\xi}\, d\mu(\xi).$$

Thus the left side of (2.1) is given by

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |e^{ix\xi} - 1|^2 |x|^{-2s-1}\, dx \right\} d\mu(\xi).$$

Making the change of variable $x \to \xi^{-1}x$ in the inner integral, one obtains the integral

$$C_s \int_{-\infty}^{\infty} |\xi|^{2s}\, d\mu(\xi), \tag{2.2}$$

with $C_s \neq 0$. But (2.2) is finite if and only if $u \in \mathcal{H}^s(\pi)$. Q.E.D.

Suppose now that G is solvable. Let $\{X_j | 1 \leqslant j \leqslant n\}$ be a Jordan-Hölder basis for $\mathfrak{g}$, i.e. $[X_j, \mathfrak{g}_{j+1}] \subseteq \mathfrak{g}_{j+1}$, where $\mathfrak{g}_j = \operatorname{span}\{X_i | i \geqslant j\}$. From the Reduction Theorem and Proposition 2.1 we obtain

THEOREM 2.2. *Let π be a unitary representation of G, π_j the restriction of π to $\{\exp tX_j | t \in \mathbf{R}\}$. Then for any $s > 0$,*

$$\mathcal{H}^s(\pi) = \bigcap_{j=1}^{n} \mathcal{H}^s(\pi_j). \tag{2.3}$$

In particular, if $0 < s < 1$, *then* $u \in \mathcal{H}^s(\pi)$ *if and only if*

$$\|u\| + \sum_{j=1}^{n} \left\{ \int_{-\infty}^{\infty} \|\pi(\exp tX_j)u - u\|^2 |t|^{-2s-1}\, dt \right\}^{1/2} < \infty, \tag{2.4}$$

and (2.4) *defines an equivalent norm on* $\mathcal{H}^s(\pi)$.

REMARK. For integral s, (2.3) was proved in [4, Theorem 5.2] for arbitrary G and any basis $\{X_j\}$ for $\mathfrak{g}$.

EXAMPLE. Let G be the "$ax + b$" group of affine transformations of $\mathbf{R}$. $\mathfrak{g}$ has a basis X_1, X_2, with $[X_1, X_2] = X_2$. Let π be the irreducible representation of G on $\mathcal{H} = L^2(\mathbf{R}; dx)$ such that $X_1 \to d/dx$ and $X_2 \to$ Multiplication by ie^x. Then for $s > 0$,

$$\mathcal{H}^s(\pi_1) = \text{usual } L^2 \text{ Sobolev space},$$

$$\mathcal{H}^s(\pi_2) = \{f \in L^2 | e^{sx}f(x) \in L^2\},$$

and $\mathcal{H}^s(\pi) = \mathcal{H}^s(\pi_1) \cap \mathcal{H}^s(\pi_2)$, with equivalent norm

$$\|f\|_s^2 = \int_{-\infty}^{\infty} e^{2sx} |f(x)|^2\, dx + \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} |\hat{f}(\xi)|^2\, d\xi$$

($\hat{f}$ = Fourier transform of f).

COROLLARY 2.3 (HYPOTHESES OF THEOREM 2.2). *Let m be a nonnegative integer and $0 < \alpha < 1$. Suppose $u \in \mathcal{H}(\pi)$ is such that for all $v \in \mathcal{H}(\pi)$ and $1 \leqslant j \leqslant n$, the functions $\phi(t) = (\pi(\exp tX_j)u, v)$ are of class C^m on $\mathbf{R}$, with $\phi^{(m)}$ Lipschitz continuous of exponent α. Then $u \in \mathcal{H}^{m+s}(\pi)$ for all $s < \alpha$.*

PROOF. By Corollary 5.1 of [4] we have $u \in \mathcal{H}^m$. Set $w = \pi_{-\infty}(X_j^m)u$, where $\pi_{-\infty}$ denotes the representation of $U(\mathfrak{g})$ on the space $\mathcal{H}^{-\infty}(\pi)$ of continuous, conjugate-linear functionals on $\mathcal{H}^{\infty}(\pi)$. Then $w \in \mathcal{H}^0$, and for any $v \in \mathcal{H}$, one has $(\pi(\exp tX_j)w, v) = \phi^{(m)}(t)$, where $\phi(t) = (\pi(\exp tX_j)u, v)$. The Lipschitz continuity of $\phi^{(m)}$ and the uniform boundedness principle imply that

$$\|\pi(\exp tX_j)w - w\| \leqslant C|t|^\alpha$$

for some constant C and $t \neq 0$. By (2.1) we conclude that $w \in \mathcal{H}^s(\pi_j)$, so that $u \in \mathcal{H}^{m+s}(\pi_j)$ for $1 \leqslant j \leqslant n$. Apply Theorem 2.2.

COROLLARY 2.4 (HYPOTHESES OF THEOREM 2.2). *Suppose $T \in U_{2m}(\mathfrak{g})$ is elliptic, $u \in \mathcal{H}(\pi)$, and $\pi_{-\infty}(T)u \in \mathcal{H}^k(\pi)$ for some nonnegative integer k. Then $u \in \mathcal{H}^{k+2m-\varepsilon}(\pi)$ for all $\varepsilon > 0$.*

PROOF. Let $v \in \mathcal{H}$. Set $w = \pi_{-\infty}(T)u$, $\phi(g) = (\pi(g)u, v)$, and $\psi(g) = (\pi(g)w, v)$. Then $\psi \in C^k(G)$, and ϕ is a weak solution to the equation $T\phi = \psi$ on G. By classical elliptic regularity theory [1], $\phi \in C^{k+2m-1}$, with all derivatives of order $k + 2m - 1$ of ϕ being Lipschitz continuous of exponent α, for any $\alpha < 1$. Apply Corollary 2.3.

REMARK. Under the hypothesis that T be associated with a *Hermitian elliptic form* in $U_m(\mathfrak{g}) \otimes U_m(\mathfrak{g})$, Corollary 2.4 holds with $\varepsilon = 0$ and any real number k (cf. [4], [5]).

3. Laplacians on Lie groups. Let $\{X_j\}$ be a basis for $\mathfrak{g}$, and set $\Delta = \sum X_j^2$, acting as a left-invariant differential operator on G. Call Δ a *Laplacian* on G. For $t > 0$, let $p_t(x)$ be the fundamental solution for the heat equation $u_t = \Delta u$ on G. Recall that a nonnegative Borel function ϕ on G is *submultiplicative* if $\phi(xy) \leqslant \phi(x)\phi(y)$. For example, $\phi(x) = \|\pi(x)\|$ is submultiplicative, for any Banach space representation π of G.

LEMMA 3.1. *Let ϕ be submultiplicative on G. There are constants $M, \omega > 0$ so that*

$$\int_G p_t(x)\phi(x)\,dx \leqslant Me^{\omega t}$$

for all $t > 0$.

PROOF. See [8, §4].

REMARK. The notion of "Bessel potential", familiar in the case $G = \mathbf{R}^n$, can be defined in general using a Laplacian Δ. Indeed, the operator $(\lambda - \Delta)^{-s}$ is bounded on $L^2(G; dx)$ for $\operatorname{Re} s > 0$, $\operatorname{Re}\lambda > 0$, and acts by right convolution with the function

$$J_\lambda^s(x) = \Gamma(s)^{-1}\int_0^\infty p_t(x)e^{-\lambda t}t^{s-1}\,dt.$$

From Lemma 3.1 one has

$$\int_G \phi(x)|J_\lambda^s(x)|\,dx \leqslant M_s(\operatorname{Re}\lambda - \omega)^{-\operatorname{Re}(s)}$$

if $\operatorname{Re}\lambda > \omega$. Thus for any Banach space representation π of G, the resolvent of the semigroup $\pi(p_t)$ is $\pi(J_\lambda^1)$. Furthermore, when π is unitary, then for any $\lambda > 0$, $s > 0$, one has

$$\mathcal{H}^s(\pi) = \operatorname{Range}\pi(J_\lambda^s)$$

(cf. [4]).

4. Noncommutative interpolation. We recall some results about interpolation spaces associated with semigroups of operators ([2], [6]). Let E be a Banach space, with norm $\|\cdot\|_E$. For $1 \leqslant p \leqslant \infty$, let $L_*^p(E)$ be the space of all strongly measurable functions $t \to u(t)$ from $(0, \infty)$ to E such that

$$\|u\|_{p,E}^p = \int_0^\infty \|u(t)\|_E^p \frac{dt}{t} < \infty.$$

(For $p = \infty$, $\|u\|_\infty = \operatorname{ess\,sup}\|u(t)\|_E$.)

Suppose that $F \subset E$ is another Banach space, continuously embedded in E. If $0 < \theta < 1$, define $(F; E)_{\theta,p}$ to be the subspace of all $x \in E$ which can be written as $x = u_0(t) + u_1(t)$, with $t^{-\theta}u_0 \in L_*^p(F)$ and $t^{1-\theta}u_1 \in L_*^p(E)$. Set

$$\|x\|_{\theta,p} = \inf\left\{\|t^{-\theta}u_0\|_{p,F}, \|t^{1-\theta}u_1\|_{p,E}\right\},$$

the inf being taken over all such pairs u_0, u_1.

EXAMPLE. Assume $F = \mathfrak{D}(A)$, where A is a closed operator on E and $-A$ generates a strongly continuous semigroup e^{-tA}. By adding a large positive constant to A, we may assume that $(A + t)^{-1}$ exists as a bounded operator on E for $t \geqslant 0$. Then $x \in (\mathfrak{D}(A); E)_{\theta,p}$ if and only if $t^{1-\theta}A(A + t)^{-1}x \in L_*^p(E)$, and the norm $\|x\|_{\theta,p}$ is equivalent to

$$\|x\|_E + \left\{\int_0^\infty \|t^{1-\theta}A(A + t)^{-1}x\|_E^p \frac{dt}{t}\right\}^{1/p}. \tag{4.1}$$

In particular, if E is a Hilbert space and $A \geqslant I$ is selfadjoint, then by (4.1) one sees that

$$(\mathfrak{D}(A); E)_{\theta,2} = \mathfrak{D}(A^{1-\theta}) \tag{4.2}$$

for $0 < \theta < 1$, with equivalent norm $\|A^{1-\theta}u\|_E$.

Suppose now that A and B are closed operators on E, with $-A$ and $-B$ each generating strongly continuous semigroups. Assume that there are constants $C, \omega > 0$ with

$$e^{-tA}: \mathfrak{D}(B) \to \mathfrak{D}(B) \quad \text{for } t > 0. \tag{4.3}$$

$$\|Be^{-tA}x\|_E \leqslant Ce^{\omega t}\{\|x\|_E + \|Bx\|_E\} \tag{4.4}$$

for $x \in \mathfrak{D}(B)$ and $t > 0$.

THEOREM 4.1 (GRISVARD [7]). *Set* $F = \mathfrak{D}(A) \cap \mathfrak{D}(B)$. *Then for* $0 < \theta < 1$ *and* $1 \leqslant p \leqslant \infty$, *one has*

$$(F; E)_{\theta,p} = (\mathfrak{D}(A); E)_{\theta,p} \cap (\mathfrak{D}(B); E)_{\theta,p}.$$

REMARK. In [7] this result is stated with $Ce^{\omega t}$ replaced by C in (4.4). This can be achieved by replacing A by $A + \omega I$, of course, without changing $\mathfrak{D}(A)$.

5. Proof of the Reduction Theorem. Take Laplacians Δ_H and Δ_K on H and K, respectively, and denote by p_t^H and p_t^K the corresponding fundamental solutions to the heat equations on H and K. Let $\sigma = \pi|_H$, $\tau = \pi|_K$, and take $A =$ closure of $\sigma_\infty(-\Delta_H)$, $B =$ closure of $\tau_\infty(-\Delta_K)$, where for any representation π of G, π_∞ denotes the representation of $U(\mathfrak{g})$ on $\mathcal{H}^\infty(\pi)$. Then $-A$ and $-B$ are the generators of the semigroups $\sigma(p_t^H)$ and $\tau(p_t^K)$, respectively. Furthermore, for $s > 0$,

$$\mathcal{H}^s(\sigma) = \mathfrak{D}(A^{s/2}), \qquad \mathcal{H}^s(\tau) = \mathfrak{D}(B^{s/2}) \tag{5.1}$$

and by [**4**, Theorem 5.2],

$$\mathcal{H}^2(\pi) = \mathfrak{D}(A) \cap \mathfrak{D}(B).$$

Set $E = \mathcal{H}(\pi)$, $F = \mathcal{H}^2(\pi)$. To verify (4.3) in this situation, it suffices by (5.1) to show that $\sigma(p_t^H)$ leaves $\mathcal{H}^2(\tau)$ invariant. But since K is a normal subgroup of G, $\mathcal{H}^2(\tau)$ is clearly invariant under $\pi(g)$, $g \in G$, and G acts continuously on $\mathcal{H}^2(\tau)$, with $\|\tau(g)v\|_{\mathcal{H}^2(\tau)} \leqslant \phi(g)\|v\|_{\mathcal{H}^2(\tau)}$, where $\phi(g) = \|\mathrm{Ad}(g^{-1})|_{U_2(\mathfrak{k})}\|$. By Lemma 3.1 it is clear that the integral

$$\int_H p_t^H(h)\pi(h)v\, dh = \sigma(p_t^H)v$$

converges absolutely in the norm of $\mathcal{H}^2(\tau)$, and

$$\|\sigma(p_t^H)v\|_{\mathcal{H}^2(\tau)} \leqslant Ce^{\omega t}\|v\|_{\mathcal{H}^2(\tau)},$$

for $v \in \mathcal{H}^2(\tau)$. By (5.1) this also establishes (4.4). Taking $p = 2$ in Theorem 4.1, we conclude by (5.1) and (4.2) that

$$\mathcal{H}^s(\pi) = \mathcal{H}^s(\sigma) \cap \mathcal{H}^s(\tau) \tag{5.2}$$

for $0 < s < 2$.

To establish (5.2) in general, we observe that by Theorem 5.2 of [**4**], $u \in \mathcal{H}^{2m+s}(\pi)$ if and only if $\pi_{-\infty}(R)u \in \mathcal{H}^2(\pi)$, where $R = \sum X_j^{2m}$ (m a positive integer, $\{X_j\}$ a basis for $\mathfrak{g}$). Choosing $\{X_j\}$ as the union of bases for $\mathfrak{h}$ and $\mathfrak{k}$, we see that $u \in \mathcal{H}^{2m+s}(\pi)$ if and only if $\pi_{-\infty}(S)u \in \mathcal{H}^s(\pi)$ and $\pi_{-\infty}(T)u \in \mathcal{H}^s(\pi)$ for all $S \in U_{2m}(\mathfrak{h})$ and $T \in U_{2m}(\mathfrak{k})$. Hence if $0 < s < 2$ and $u \in \mathcal{H}^{2m+s}(\sigma) \cap \mathcal{H}^{2m+s}(\tau)$, it follows by (5.2) that $u \in \mathcal{H}^{2m+s}(\pi)$. The opposite inclusion is evident by the monotonicity property of interpolation, which completes the proof.

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