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APPLICATIONS OF THE u-CLOSURE OPERATOR

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To Professor George H. Butcher on his 60th birthday

ABSTRACT. Let $\text{cl}_u(A)$ be the u-closure of a subset A of a space. We prove that a space is compact if and only if for each upper-semicontinuous multifunction λ on the space, the multifunction μ defined on the space by $\mu(x) = \text{cl}_u(\lambda(x))$ assumes a maximal value under set inclusion. We also prove that in a Urysohn-closed space any two subsets with disjoint u-closures are separated by disjoint open subsets. The quotient space induced by identifying those points with identical u-closures is investigated and shown to be T_0.

Introduction. In $[\mathbf{H}_1]$ the notion of *u-convergence* of a filterbase was introduced and utilized to study Urysohn-closed and minimal Urysohn topological spaces in terms of arbitrary filterbases. The notion of u-convergence leads naturally to the concept of *u-closure* of a subset A ($\text{cl}_u(A)$) of a space which was employed in $[\mathbf{J}_1]$ to obtain characterizations of Urysohn-closed and minimal Urysohn spaces. In this paper we (1) define certain subsets of a space in terms of cl_u, relate these subsets to others which have recently been studied and establish some decomposition and separation properties for these subsets, (2) show that a space X is compact if and only if for each upper-semicontinuous multifunction λ on X, the multifunction μ defined on X by $\mu(x) = \text{cl}_u(\lambda(x))$ assumes a maximal value under set inclusion, (3) use the result in (2) to prove a result for the compact-open topology and (4) study the quotient topology induced by identifying those points with identical u-closures, using the above results to show that this quotient space is always T_0 and to gather other information about this space.

Results. Let X be a space, $A \subset X$, $x \in X$ and let Ω be a filterbase on X. We let $\text{cl}(A)$ and $\text{ad}\,\Omega$ represent the closure of A and adherence of Ω respectively. We denote by $\Sigma(A)$ the collection of open neighborhoods of A and by $\Lambda(A)$ the collection of open sets which contain closed neighborhoods of A. (If $A = \{x\}$, we write $\Sigma(x)$ or $\Lambda(x)$.) The *u-closure* of A is $\{v \in X\colon \text{each } V \in \Lambda(v) \text{ satisfies } A \cap \text{cl}(V) \neq \varnothing\}$, and the *u-adherence* of Ω ($\text{ad}_u\,\Omega$) is $\bigcap_\Omega \text{cl}_u(F)$. The statements in our first proposition will be useful in the sequel. The proof of the proposition is straightforward and is omitted.

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PROPOSITION 1. *Let X be a space, let $A \subset X$, let Ω be a filterbase on X and let $x, y \in X$.*

(a) *The equations*

$$\mathrm{cl}_u(A) = \bigcap_{\Sigma(A)} \mathrm{cl}_u(V) = \bigcap_{\Lambda(A)} \mathrm{cl}(W)$$

and

$$\mathrm{ad}_u\, \Omega = \mathrm{ad}_u \bigcup_{\Omega} \Sigma(F) = \mathrm{ad} \bigcup_{\Omega} \Lambda(F)$$

hold.

(b) *If $x \in \mathrm{cl}_u(y)$ then $y \in \mathrm{cl}_u(x)$.*

A net g *u-converges* to x in a space if g is eventually in $\mathrm{cl}(W)$ for each $W \in \Lambda(x)$ [$\mathbf{H_1}$]. The question is raised in [$\mathbf{H_1}$] as to how an open filterbase Ω can be constructed from a net g with the property that some subnet of g u-converges to x if and only if $x \in \mathrm{ad}\, \Omega$. In our first theorem this question is answered by appeal to Proposition 1(a) and standard methods.

THEOREM 1. *Let X be a space and let (g, D) be a net in X. For each $n \in D$, let $S(n) = \{g(k): k \geqslant n\}$. Then $\Omega(g) = \bigcup_D \Lambda(S(n))$ is an open filterbase on X, and $x \in \mathrm{ad}\, \Omega(g)$ if and only if some subnet of g u-converges to x.*

We recall [V] that the *θ-closure* of a subset A ($\mathrm{cl}_\theta(A)$) of a space is $\{x \in X$: Each $V \in \Sigma(x)$ satisfies $A \cap \mathrm{cl}(V) \neq \varnothing\}$ and that $\bigcap_\Omega \mathrm{cl}_\theta(F)$ is the *θ-adherence* of a filterbase Ω on a space. It is readily seen that $\mathrm{cl}_u(V) = \mathrm{cl}_\theta(\mathrm{cl}(V))$ for any open subset V of a space. Let X be a space and let $A \subset X$. A is *quasi H-closed* (*QHC*) *relative to X* if each filterbase Ω on A satisfies $A \cap \mathrm{ad}_\theta\, \Omega \neq \varnothing$ [$\mathbf{H_2}$]. We say simply that A is *quasi H-closed* (*QHC*) if A is QHC relative to A; we define A to be *quasi Urysohn-closed* (*QUC*) *relative to X* if each filterbase Ω on A satisfies $A \cap \mathrm{ad}_u\, \Omega \neq \varnothing$. It is known that A is $U(i)$ in the sense of Scarborough if and only if A is QUC relative to A. It is clear from the definitions and the relationship between cl_u and cl_θ on open subsets that A is QUC relative to X if A is QHC relative to X. Propositions 2 and 3 are stated without proof since the proofs are similar to those of the analogous results for QHC relative subsets.

PROPOSITION 2. *The following statements are equivalent for a space X and $A \subset X$.*

(a) *A is QUC relative to X.*

(b) *Each open filterbase Ω on A satisfies $A \cap \mathrm{ad}_u\, \Omega \neq \varnothing$.*

(c) *$A \cap \mathrm{ad} \bigcup_\Omega \Lambda(F) \neq \varnothing$ is satisfied for each filterbase Ω on A.*

(d) *For each Urysohn open cover* [$\mathbf{H_1}$], *Δ, of A by open subsets of X, some finite $\Delta^* \subset \Delta$ satisfies $A \subset \bigcup_{\Delta^*} \mathrm{cl}(V)$.*

(e) *Each net g in A has a subnet which u-converges to some point in A.*

(f) *Each filterbase Ω on X satisfying $F \cap C \neq \varnothing$ for each regular-closed subset which contains A also satisfies $A \cap \mathrm{ad}_u\, \Omega \neq \varnothing$.*

(g) *Each base for an ultrafilter on A u-converges* [$\mathbf{H_1}$] *to some point in A.*

We say that a subset A of a space is *u-closed* if $\mathrm{cl}_u(A) = A$ and *θ-closed* if $\mathrm{cl}_\theta(A) = A$.

PROPOSITION 3. *Let X be a space and let $A \subset X$. If A is QUC relative to X then* $\mathrm{cl}(A)$ *is QUC relative to X. If X is $U(i)$ and A is u-closed then A is QUC relative to X.*

It is known [J_2] that if X is QHC then $\mathrm{cl}_\theta(A)$ is QHC relative to X for each $A \subset X$. Our next theorem improves this result.

THEOREM 2. *If X is QHC and $A \subset X$ then* $\mathrm{cl}_u(A)$ *is QHC relative to X.*

PROOF. Let Ω be a filterbase on $\mathrm{cl}_u(A)$. If $W \in \Lambda(A)$, then $\mathrm{cl}_u(A) \subset \mathrm{cl}(W)$ and, consequently, $V \cap W \neq \varnothing$ is satisfied for all $V \in \Sigma(F)$, $F \in \Omega$ and $W \in \Lambda(A)$. Hence $\Omega^* = \{V \cap W\colon V \in \bigcup_\Omega \Sigma(F),\ W \in \Lambda(A)\}$ is an open filterbase on X. Hence $\varnothing \neq \mathrm{ad}\, \Omega^* \subset \mathrm{ad}_\theta\, \Omega \cap \mathrm{cl}_u(A)$. The proof is complete.

We leave the following problem open.

Problem. Prove or disprove that $\mathrm{cl}_u(A)$ is QUC relative to X for each subset A of a $U(i)$ space.

In the following theorem we present an interesting set inclusion relation between the θ-closure of a QHC relative subset and the u-closures of the points of the set.

THEOREM 3. *If A is QHC relative to a space X then* $\mathrm{cl}_\theta(A) \subset \bigcup_A \mathrm{cl}_u(x)$.

PROOF. Let $y \in \mathrm{cl}_\theta(A)$. Then $A \cap \mathrm{cl}(V) \neq \varnothing$ is satisfied for all $V \in \Sigma(y)$. Hence $\varnothing \neq A \cap \bigcap_{\Sigma(y)} \mathrm{cl}_\theta(\mathrm{cl}(V)) = A \cap \mathrm{ad}_u\, \Sigma(y) = A \cap \mathrm{cl}_u(y)$. By Proposition 1(b), $y \in \bigcup_A \mathrm{cl}_u(x)$. The proof is complete.

A subset A of a space X is *θ-rigid* if each filterbase Ω on X satisfying $F \cap \mathrm{cl}(V) \neq \varnothing$ for all $F \in \Omega$ and $V \in \Sigma(A)$ also satisfies $A \cap \mathrm{ad}_\theta\, \Omega \neq \varnothing$ **[E-J]**; we define A to be *u-rigid* if each filterbase Ω on X satisfying $F \cap \mathrm{cl}(W) \neq \varnothing$ for each $W \in \Lambda(A)$ and $F \in \Omega$ also satisfies $A \cap \mathrm{ad}_u\, \Omega \neq \varnothing$. The following characterization result is stated without proof; $\mathrm{int}(A)$ represents the interior of a subset A of a space.

PROPOSITION 4. *The following statements are equivalent for a space X and $A \subset X$.*

(a) *A is u-rigid.*

(b) *For each Urysohn open cover Δ of A by open subsets of X some finite $\Delta^* \subset \Delta$ satisfies* $A \subset \mathrm{int}(\bigcup_{\Delta^*} \mathrm{cl}(V))$.

(c) *Each open filterbase Ω on X satisfying $V \cap W \neq \varnothing$ for all $V \in \Omega$ and $W \in \Lambda(A)$ also satisfies* $A \cap \mathrm{ad}_u\, \Omega \neq \varnothing$.

(d) *If a net in X is frequently in* $\mathrm{cl}(W)$ *for each $W \in \Lambda(A)$ then some subnet of the net u-converges to some point in A.*

(e) $A \cap \mathrm{ad} \bigcup_\Omega \Lambda(F) \neq \varnothing$ *is satisfied by any filterbase Ω on X such that $V \cap W \neq \varnothing$ is satisfied for all $V \in \bigcup_\Omega \Sigma(F)$ and $W \in \Lambda(A)$.*

(f) *Each base β for an ultrafilter on X satisfying $B \cap \mathrm{cl}(W) \neq \varnothing$ for all $B \in \beta$ and $W \in \Lambda(A)$ u-converges to a point in X.*

Since there are Urysohn-closed spaces which are not H-closed, it is clear that there are u-rigid subsets which are not θ-rigid. The next result shows that θ-rigid subsets are u-rigid.

THEOREM 4. *A θ-rigid subset of a space is u-rigid.*

PROOF. The proof follows from Proposition 4(b) and the recollection that a subset A of a space X is θ-rigid if and only if for each cover Δ of A by open subsets of X some finite $\Delta^* \subset \Delta$ satisfies $A \subset \operatorname{int}(\bigcup_{\Delta^*} \operatorname{cl}(V))$.

In our next theorem we give a decomposition result for u-rigid subsets.

THEOREM 5. *If A is a u-rigid subset of a space then* $\operatorname{cl}_u(A) = \bigcup_A \operatorname{cl}_u(x)$.

PROOF. Let $y \in \operatorname{cl}_u(A)$. Then $y \in \operatorname{cl}(W)$ for each $W \in \Lambda(A)$. Hence $A \cap \operatorname{cl}_u(y) \neq \varnothing$ and, consequently, $y \in \bigcup_A \operatorname{cl}_u(x)$. The proof of the reverse inclusion is obvious.

COROLLARY 1. *If A is a θ-rigid subset of a space then* $\operatorname{cl}_u(A) = \bigcup_A \operatorname{cl}_u(x)$.

Our next three results provide information on separation of subsets of spaces by disjoint open sets.

THEOREM 6. *Two subsets of a $U(i)$ space with disjoint u-closures are separated by disjoint open subsets.*

PROOF. Let X be $U(i)$ and let A, B be subsets of X satisfying $\operatorname{cl}_u(A) \cap \operatorname{cl}_u(B) = \varnothing$. If all $V \in \Sigma(A)$ and $W \in \Sigma(B)$ satisfy $V \cap W \neq \varnothing$ then $\Omega = \{V \cap W: V \in \Sigma(A), W \in \Sigma(B)\}$ is an open filterbase on X. Since X is $U(i)$ we have

$$\varnothing \neq \operatorname{ad}_u \Omega \subset \operatorname{ad}_u \Sigma(A) \cap \operatorname{ad}_u \Sigma(B) = \operatorname{cl}_u(A) \cap \operatorname{cl}_u(B).$$

This is a contradiction and the proof is complete.

COROLLARY 2. *Two disjoint u-closed subsets of a $U(i)$ space are separated by disjoint open subsets.*

THEOREM 7. *If A, B are subsets of a space X with A u-rigid and $A \cap \operatorname{cl}_u(B) = \varnothing$ then A and B are separated by disjoint open subsets.*

PROOF. If $\Omega = \{V \cap W: V \in \Sigma(A), W \in \Sigma(B)\}$ is a filterbase on X then, since A is u-rigid, we have $A \cap \operatorname{ad}_u(\Sigma(B)) \neq \varnothing$. The proof is complete.

Our next two results are preliminary to our characterizations of compactness in terms of cl_u and upper-semicontinuous multifunctions. A *multifunction* from a set X to a set Y is a function from X to $P(Y) - \{\varnothing\}$, where $P(Y)$ is the family of subsets of Y. If λ is a multifunction from X to Y we will write $\lambda \in M(X, Y)$ and if $A \subset X$ we write $\lambda(A)$ for $\bigcup_A \lambda(x)$; if X and Y are spaces and $x \in X$, we say that λ is *upper-semicontinuous* (*u.s.c.*) at x if for each $W \in \Sigma(\lambda(x))$ some $V \in \Sigma(x)$ satisfies $\lambda(V) \subset W$; λ is *upper-semicontinuous* (*u.s.c.*) if λ is u.s.c. at each $x \in X$. A multifunction $\lambda \in M(X, Y)$ has a *u-strongly-closed graph* if $\operatorname{ad}_u \lambda(\Sigma(x)) = \lambda(x)$ for each $x \in X$ [$\mathbf{J_1}$]. Each function with a u-strongly-closed graph has a strongly-closed graph [**H-L**].

THEOREM 8. *If* $\lambda \in M(X, Y)$ *is u.s.c. then* $\mathrm{ad}_u\, \lambda(\Sigma(x)) = \mathrm{cl}_u(\lambda(x))$.

PROOF. It is clear that $\mathrm{cl}_u(\lambda(x)) \subset \mathrm{ad}_u\, \lambda(\Sigma(x))$ and for each $W \in \Sigma(\lambda(x))$, some $V \in \Sigma(x)$ satisfies $\lambda(V) \subset W$ and thus $\mathrm{ad}_u\, \lambda(\Sigma(x)) \subset \mathrm{cl}_u(\lambda(x))$. This completes the proof.

COROLLARY 3. *A u.s.c. multifunction* λ *has a u-strongly-closed graph if and only if* λ *has u-closed point images.*

THEOREM 9. *The following statements are equivalent for a space* X.

(a) X *is compact.*

(b) *For each u.s.c. multifunction* λ *on* X *the multifunction* μ *on* X *defined by* $\mu(x) = \mathrm{cl}_u(\lambda(x))$ *assumes a maximal value under set inclusion.*

(c) *Each u.s.c. multifunction* λ *on* X *with u-closed point images assumes a maximal value under set inclusion.*

(d) *Each u.s.c. multifunction* λ *on* X *with a u-strongly-closed graph assumes a maximal value under set inclusion.*

PROOF. The equivalence of (c) and (d) follows from Corollary 3, and (c) is obviously implied by (b). To establish that (a) implies (b) let $\Omega = \{\mu(x): x \in X\}$ be ordered by set inclusion and let Ω^* be a nonempty chain in Ω. For each y such that $\mu(y) \in \Omega^*$ let $F(y) = \{x \in X: \mu(y) \subset \mu(x)\}$. Then $\{F(y)\}$ is a filterbase on the compact space X. If $\mu(y) \in \Omega^*$ let $v \in \mathrm{cl}(F(y))$ and let $W \in \Sigma(\lambda(v))$. Some $V \in \Sigma(v)$ satisfies $\lambda(V) \subset W$. If $q \in V \cap F(y)$ then $\mu(y) \subset \mu(q) = \mathrm{cl}_u(\lambda(q)) \subset \mathrm{cl}_u(W)$. Hence $\mu(y) \subset \mu(v)$, $v \in F(y)$ and $F(y)$ is closed. Let $q \in \bigcap F(y)$. Then $\mu(q)$ is an upper bound for Ω^*. By Zorn's Lemma, Ω has a maximal element. To complete the proof, we will verify that (a) is implied by (c). If X is not compact there is a net g in X with an ordinal D as its index set and no convergent subnet. Let D have the order topology and for each $k \in D$ let $V(k) = X - \mathrm{cl}(\{g(j): j \geqslant k\})$. Then $\{V(k): k \in D\}$ is an increasing open cover of X with no finite subcover. Define $\lambda \in M(X, D)$ by $\lambda(x) = \{j \in D: j < k(x)\}$ where $k(x)$ is the first element k of D with $x \in V(k)$. Since D with the order topology is regular and $\lambda(x)$ is closed for each x then $\mu(x) = \lambda(x)$ for each x. If $W \in \Sigma(\lambda(x))$ and $y \in V(k(x))$ then $k(y) \leqslant k(x)$ so that $\lambda(y) \subset \lambda(x) \subset W$. Therefore $\lambda(V(k(x))) \subset W$ and λ is u.s.c. Since μ clearly assumes no maximal value with respect to set inclusion we see that (c) fails. This completes the proof.

In a Urysohn space the u-closure of each point is trivially compact and maximal in the set of u-closures of points ordered by inclusion. We may use the results in Theorem 9 to prove that in any space the u-closures of points satisfy a maximality condition when the u-closure of some point is compact.

THEOREM 10. *Let* Y *be a space and let* $y_0 \in Y$ *with* $\mathrm{cl}_u(y_0)$ *compact. Then there is a* $y \in Y$ *such that* (a) $\mathrm{cl}_u(y_0) \subset \mathrm{cl}_u(y)$ *and* (b) $\mathrm{cl}_u(y)$ *is maximal in the set of u-closures of points when this set is ordered by inclusion.*

PROOF. Let $X = \{y \in Y: \mathrm{cl}_u(y_0) \subset \mathrm{cl}_u(y)\}$. For each $y \in X$ we have $y \in \mathrm{cl}_u(y_0)$ from Proposition 1(b). Moreover, if $v \in \mathrm{cl}(X)$ and $W \in \Sigma(v)$ then some $y \in W$

satisfies $\mathrm{cl}_u(y_0) \subset \mathrm{cl}_u(y) \subset \mathrm{cl}_u(W)$. Hence $\mathrm{cl}_u(y_0) \subset \mathrm{cl}_u(v)$ and X is closed in Y. Therefore X is a compact subset of Y and since the identity function from X to Y is u.s.c. the proof may be completed by appeal to the fact that (a) implies (b) in Theorem 9.

The proofs of the following easily established corollaries are omitted.

COROLLARY 4. *If Y is compact then for each $y_0 \in Y$ there is a $y \in Y$ such that* (a) $\mathrm{cl}_u(y_0) \subset \mathrm{cl}_u(y)$ *and* (b) $\mathrm{cl}_u(y)$ *is maximal in the set of u-closures of points when this set is ordered by inclusion.*

COROLLARY 5. *If Y is a regular space and $y_0 \in Y$ then there is a $y \in Y$ such that* (a) $y_0 \in \mathrm{cl}(y)$ *and* (b) $\mathrm{cl}(y)$ *is maximal in the set of closures of points when this set is ordered by inclusion.*

If X is a space and F is a collection of functions from X to a space Y with F having the compact-open topology it is known that for each nonempty compact $A \subset X$ the multifunction $H_A \in M(F, Y)$ defined by $H_A(g) = g(A)$ is u.s.c. so we may prove the following corollary to Theorem 9.

COROLLARY 6. *If X and Y are spaces and F is a compact family of functions from X to Y with the compact-open topology then for each nonempty compact $A \subset X$ there is a function g from X to Y such that* $\mathrm{cl}_u(g(A))$ *is maximal with respect to set inclusion.*

The quotient space induced by identifying those points of a given space with identical closures has been extensively studied. In the last results in this paper we initiate the study of the quotient space induced by identifying those points of a given space with identical u-closures (i.e. we say that x is equivalent to y if $\mathrm{cl}_u(x) = \mathrm{cl}_u(y)$). For $A \subset X$, let $u[A]$ ($u[x]$ if $A = \{x\}$) represent the saturation of A by the equivalence relation (i.e. $u[A] = \{y \in X\colon y$ is equivalent to some $x \in A\}$). A is saturated with the relation if $u[A] = A$. The next proposition follows from previous results.

PROPOSITION 5. *The following properties hold for a topological space X.*

(a) *Each $x \in X$ satisfies* $u[\mathrm{cl}_u(x)] = \mathrm{cl}_u(x)$.

(b) *Each u-rigid $A \subset X$ satisfies* $u[\mathrm{cl}_u(A)] = \mathrm{cl}_u(A)$.

(c) *Each θ-rigid $A \subset X$ satisfies* $u[\mathrm{cl}_u(A)] = \mathrm{cl}_u(A)$.

(d) *Each $A \subset X$ satisfies* $u[A] \subset \mathrm{cl}_u(A)$.

(e) *If $A \in \Lambda(B)$ in X then* $u[B] \subset \mathrm{cl}(A)$.

(f) *For each $x \in X$,* $\bigcap_{\mathrm{cl}_u(x)} \mathrm{cl}_u(v) = \{y \in X\colon \mathrm{cl}_u(x) \subset \mathrm{cl}_u(y)\}$.

(g) *For $x, y \in X$ the relations* (i) $y \in \mathrm{cl}_u(x)$, (ii) $u[y] \cap \mathrm{cl}_u(x) \neq \varnothing$, (iii) $u[x] \cap \mathrm{cl}_u(y) \neq \varnothing$, (iv) $u[x] \subset \mathrm{cl}_u(y)$ *and* (v) $u[y] \subset \mathrm{cl}_u(x)$ *are equivalent.*

PROOF. For the proof of (a) let $y \in u[\mathrm{cl}_u(x)]$. There is a $v \in \mathrm{cl}_u(x)$ such that $y \in u[v]$. From Proposition 1(b), $x \in \mathrm{cl}_u(v)$ and since $\mathrm{cl}_u(v) = \mathrm{cl}_u(y)$ we obtain $y \in \mathrm{cl}_u(x)$. Hence the proof of (a) is complete since $\mathrm{cl}_u(x) \subset u[\mathrm{cl}_u(x)]$ from a general property of equivalence relations. The proof that (b) holds follows directly from (a), Theorem 5 and the fact that $u[\bigcup_\Omega F] = \bigcup_\Omega u[F]$ for any family Ω of subsets of X. It is obvious from Corollary 1 and (b) that (c) holds. To prove (d) we note that

$$u[A] = \bigcup_A u[x] \subset \bigcup_A u[\mathrm{cl}_u(x)] = \bigcup_A \mathrm{cl}_u(x) \subset \mathrm{cl}_u(A)$$

for any $A \subset X$. We see that (e) follows from (d) and the readily established fact that $\mathrm{cl}_u(B) \subset \mathrm{cl}(A)$ when $A \in \Lambda(B)$. Similar methods may be utilized to establish (f) and (g). The proofs are omitted.

Let $X(\mathrm{mod}\, u)$ represent the quotient space induced on X by the equivalence relation obtained. The following result is interesting.

THEOREM 11. *$X(\mathrm{mod}\, u)$ is T_0 for any space X.*

PROOF. Suppose $x, y \in X$ with $u[x] \neq u[y]$. Without loss of generality let $v \in \mathrm{cl}_u(x) - \mathrm{cl}_u(y)$. Then $y \notin \mathrm{cl}_u(v)$ and, consequently, $u[y] \cap \mathrm{cl}_u(v) = \varnothing$ from Proposition 5(g). Hence $u[y] \subset X - \mathrm{cl}_u(v)$ and $u[x] \subset \mathrm{cl}_u(v)$. Since $X - \mathrm{cl}_u(v)$ is an open subset of X saturated with the relation we conclude that $X(\mathrm{mod}\, u)$ is T_0 and the proof is complete.

If $\mathrm{cl}_u(x)$ is maximal in the set of u-closures of points when this set is ordered by inclusion it follows that $u[x] = \{y \in X: \mathrm{cl}_u(x) \subset \mathrm{cl}_u(y)\}$ and from Proposition 5(f) we have $u[x]$ closed in X. Hence we obtain the following proposition and corollaries.

PROPOSITION 6. *If X is a space and $\mathrm{cl}_u(x)$ is maximal in the set of u-closures of points when this set is ordered by inclusion then $u[x]$ is closed in X.*

COROLLARY 7. *If X is a space and $\mathrm{cl}_u(x)$ is maximal for all $x \in X$ then $X(\mathrm{mod}\, u)$ is T_1.*

COROLLARY 8. *If X is compact then $X(\mathrm{mod}\, u)$ has at least one closed singleton.*

PROOF. Theorem 10 and Proposition 6.

We complete the paper by giving several examples in connection with the preceding results. Let $\{p(k): k = 0, 1, 2, 3, \ldots\}$ be a strictly increasing sequence of primes. Let N be the set of natural numbers and let $W = N \cup \{0\}$. For $(j, k, m) \in N \times W \times N$ let $H(j, k, m) = \{(j + [p(k)]^{-n}, m): n \in N\}$; now let $J = (W \times N) \cup \bigcup_{N \times W} H(j, k, 1) \cup \bigcup_{N \times N \times N} H(j, k, m) \cup \{(0, 0), (1, 0)\}$ with the topology generated by the aggregate of basic open sets listed below.

(a) Subsets of $J - ((W \times N) \cup \{(0, 0), (1, 0)\})$.

(b) Subsets of the form $\{(0, 0)\} \cup \bigcup_{j \geq j_0} H(j, 0, 1)$.

(c) Subsets of the form $\{(1, 0)\} \cup \bigcup_{m \geq m_0} H(j, k, m)$.

(d) Subsets of the form $\{(0, m)\} \cup \bigcup_{j \geq j_0; k \geq k_0} H(j, k, m)$.

(e) All relative open sets from the plane in $\{(k, 1): k \in N\} \cup \bigcup_{N \times W} H(j, k, 1)$.

(f) For $m > 1$ and $i \in N$, all sets of the form $A \cup B$ where A is relatively open in X from the plane about (i, m) and B is of the form $\bigcup_{j \geq j_0} H(j, i, m - 1)$.

J satisfies the following properties.

(a) J is H-closed.

(b) The set $\mathrm{cl}_u^n((0, 0))$ fails to be u-closed for each $n \in W$.

(c) $J(\mathrm{mod}\ u)$ is not T_2 even though $\mathrm{cl}_u(x)$ is maximal under set inclusion for each x. If $x \in J - [(W \times (N - \{1\})) \cup \{(0, 0)\}]$, then $\mathrm{cl}_u(x) = \{x\}$; $\mathrm{cl}_u((0, 0)) = \mathrm{cl}_u((0, 1)) = \{(0, 0)\} \cup (N \times \{2\}) \cup \{(0, 1)\} = \mathrm{cl}_u((n, 2))$ for each $n \in N$; for $m > 2$, $\mathrm{cl}_u((0, m - 1)) = \{(0, m - 1)\} \cup (N \times \{m\}) = \mathrm{cl}_u((j, m))$ for each $j \in N$. Hence $\mathrm{cl}_u(x)$ is clearly maximal for each $x \in J$. However, if $V \in \Sigma(u[(0, 0)])$ and $A \in \Sigma(u[(0, 2)])$ we see that $V \cap A \neq \varnothing$. Thus $J(\mathrm{mod}\ u)$ is not T_2.

(d) Let V be a basic open set about $(0, 0)$. Then $u[V] = V \cup (N \times \{2\}) \cup \{(0, 1)\}$ which is not open in J.

Finally, let $n = 1, 2, 3, 4$, let $A(1)$ be the set of primes larger than 9 and let $A(n)$ be the closed interval $[2n, 2n + 1]$ otherwise. For each n, let $\Omega(n)$ be the filter of finite complements on $A(n)$. Let $X = \bigcup A(n) \cup \{0, 1\}$ with the topology generated by the following open set base: $\{V \subset X\colon V$ is a usual open subset of $\bigcup_{n>2} A(n)\} \cup \{\{0\} \cup F(1) \cup F(2) \cup F(3)\colon F(n) \in \Omega(n),\ n = 1, 2, 3\} \cup \{\{1\} \cup F(3) \cup F(4)\colon F(n) \in \Omega(n), n = 3, 4\} \cup \{\{p\} \cup F\colon p \in A(1), F \in \Omega(2)\}$. Then X is compact and T_1, but $u[11]$ is not closed in X since $0 \in \mathrm{cl}(u[11]) - u[11]$.

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