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A REFINEMENT ON MICHAEL'S CHARACTERIZATION OF PARACOMPACTNESS

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ABSTRACT. It is proved that a T_1 space is paracompact (and Hausdorff) if (and only if) every well (but arbitrarily) ordered open cover has an open refinement in the form of a countable union of families, (each of which being well ordered by the naturally induced well order) the initial segments of each of which are cushioned in the corresponding initial segments of the cover, i.e., the closures of the unions of the initial segments of each of which are contained within the unions of the corresponding initial segments of the cover.

1. We attempt here to improve on Michael's result that characterizes paracompactness as the existence, for every open cover, of a σ-cushioned open refinement **[7]**; in the manner of Junnila **[3]**, Katuta **[4]**, **[5]**,[1] Mack **[6]**, Tamano **[9]**, Tamano and Vaughan **[10]**, Vaughan **[11]**, Yajima **[12]** and others, where various orders are introduced on the covers in order to restrict the working of the cushioning property in various ways. Our main theorem, Theorem 3, along with the preliminary steps, by virtue of their clearly discernible demonstrability, lays bare more than most theorems do the relationship between local finiteness, the cushioning property of Michael and our characterizations here.

2. Our starting point is an old theorem of Nagami's **[8]**.[1]

THEOREM 1. *A countably paracompact T_4 space is paracompact if* (*and only if*) *every open cover has a σ-disjoint open refinement*, *i.e.*, *if* (*and only if*) *the space is* screenable **[1]**.

In order to use this theorem, we note the following lemma.

LEMMA 2. *Given a space* $(X, \mathfrak{T})$. *An open cover* $\mathfrak{U}$ *of the space has a σ-disjoint open refinement provided there exist on* $\mathfrak{U}$ *a well ordering* $<$ *and a countable family* $\mathfrak{F}$ *of functions into* $\mathfrak{T}$ *such that*

(i) $\bigcup_{f\in\mathfrak{F}} f[\mathfrak{U}]$ *covers* X, *and*, *for all* $g \in \mathfrak{F}$, $U \in \mathfrak{U}$,

(ii) $g(U) \subset U$, *and*

(iii) $\mathrm{Cl} \bigcup \{g(V)\colon V < U\} \subset \bigcup \{f(V)\colon V < U, f \in \mathfrak{F}\}$.

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[1] I am grateful to the referee for calling my attention to these two works.

PROOF. The families

$$\mathcal{Q}(f) \equiv \{f(U) \setminus \mathrm{Cl} \bigcup \{f(V): V < U\}: U \in \mathfrak{U}\}, \qquad f \in \mathfrak{F},$$

are (clearly) disjoint collections of open sets; and the family $\mathcal{Q} \equiv \bigcup_{f \in \mathfrak{F}} \mathcal{Q}(f)$ refines $\mathfrak{U}$. It therefore suffices to prove that $\mathcal{Q}$ is a cover, which can easily be seen if we observe that for each $\xi \in X$, there is, because of (i), a $W \in \mathfrak{U}$ such that

$$\xi \in k(W), \quad \text{for some } k \in \mathfrak{F},$$

and,

$$\xi \notin f(V), \quad \text{if } V < W, \text{ for any } f \in \mathfrak{F};$$

and it follows that $\xi \in k(W) \setminus \mathrm{Cl} \bigcup \{k(V): V < W\}$. For, otherwise, $\xi \in \mathrm{Cl} \bigcup \{k(V): V < W\}$ and therefore $\xi \in \bigcup \{f(V): V < W, f \in \mathfrak{F}\}$, which is a contradiction.

REMARKS. In the hypothesis of Lemma 2 above, the range of the members of $\mathfrak{F}$ need not be $\mathfrak{T}$. So long as $\bigcup \{g(V): V \leqslant U\} \in \mathfrak{T}$, for all $g \in \mathfrak{F}$, $U \in \mathfrak{U}$, we will have the same conclusion. From Theorem 1 and Lemma 2 one obtains easily characterizations of paracompactness similar to that of Katuta's in Theorem 1.1 of **[5]**.

3. Lemma 2 and Theorem 1 together give us our main result, Theorem 3.

THEOREM 3. *A T_1 space $(X, \mathfrak{T})$ is paracompact if on every well ordered open cover $(\mathfrak{U}, <)$ of X, there can be constructed functions f^i, $i \in \mathbf{N}$, into $\mathfrak{T}$ such that*

(i) $\bigcup_{i \in \mathbf{N}} f^i[\mathfrak{U}]$ *covers X, and, for all $U \in \mathfrak{U}$, $i \in \mathbf{N}$,*

(ii) $f^i(U) \subset U$, *and*

(iii) $\mathrm{Cl} \bigcup \{f^i(V): V < U\} \subset \bigcup \{V: V < U\}$.

PROOF. $(X, \mathfrak{T})$ is normal if the hypothesis is satisfied because, if A and B are two disjoint closed sets (in X), on the two ordered open covers, $\{\sim A, \sim B\}$ and $\{\sim B, \sim A\}$ there are respectively f^i, $i \in \mathbf{N}$, and g^i, $i \in \mathbf{N}$, so that we can manufacture open sets

$$(\sim B)_i \equiv g^i(\sim B) \setminus \mathrm{Cl} \bigcup_{j<i} f^j(\sim A), \qquad i \in \mathbf{N},$$

the union of which contains A and the closure of the union of which is disjoint from B. That $(X, \mathfrak{T})$ is countably paracompact if the hypothesis is satisfied can readily be seen if one notes Dowker's characterization of the property designated (d) in Theorem 2 of **[2]**.

Let $\mathfrak{W}$ be an open cover of X. We are to show that on $\mathfrak{W}$ there are a well ordering $\ll$ and a countable family $\mathcal{G}$ of functions into $\mathfrak{T}$ such that $\bigcup_{g \in \mathcal{G}} g[\mathfrak{W}]$ covers X and for every $g \in \mathcal{G}$,

(i) $g(W) \subset W$, for all $W \in \mathfrak{W}$, and

(ii) $\mathrm{Cl} \bigcup \{g(V): V \ll W\} \subset \bigcup \{f(V): V \ll W, f \in \mathcal{G}\}$, for all $W \in \mathfrak{W}$.

Let $\ll$ be *any* well order on $\mathfrak{W}$. For all $i \in \mathbf{N}$, let $W^i \equiv W$, $W \in \mathfrak{W}$, and $\mathfrak{W}^i \equiv \mathfrak{W}$. Suppose for some $m \in \mathbf{N}$ and $\lambda \in \mathbf{N}^m$, open W^λ has been defined for every $W \in \mathfrak{W}$ such that

(a) $W^\lambda \subset W$ and

(b) $\mathcal{W}^\lambda \equiv \{V^\lambda : V \in \mathcal{W}\}$ is a cover of X.

We impose such a well order $<^\lambda$ on $\mathcal{W}^\lambda$ that $V^\lambda <^\lambda U^\lambda \Leftrightarrow V \ll U$, for all $V, U \in \mathcal{W}$; and we choose, as we are entitled to choose, a sequence of functions $f^{\lambda,i}$ on $(\mathcal{W}^\lambda, <^\lambda)$ satisfying (i)–(iii) in the hypothesis with respect to $(\mathcal{W}^\lambda, <^\lambda)$, and we define, for all $W \in \mathcal{W}$, $i \in \mathbf{N}$,

$$W^{\lambda,i} \equiv W \setminus \mathrm{Cl} \bigcup \{f^{\lambda,i}(V^\lambda) : V \ll W\}.$$

Clearly it is true that $W^{\lambda,i} \subset W$, for all $W \in \mathcal{W}$. That $\mathcal{W}^{\lambda,i} \equiv \{W^{\lambda,i} : W \in \mathcal{W}\}$ is a cover follows from the facts that

$$\mathrm{Cl} \bigcup \{f^{\lambda,i}(V^\lambda) : V \ll W\} \subset \bigcup \{V^\lambda : V \ll W\} \subset \bigcup \{V : V \ll W\}$$

and that $\mathcal{W}$ is a cover. We therefore have defined, for all $n \in \mathbf{N}$ and all $\mu \in \mathbf{N}^n$, open sets W^μ, for all $W \in \mathcal{W}$, and well ordered open covers $(\mathcal{W}^\mu, <^\mu)$ and sequences of functions $f^{\mu,i}$ (on $(\mathcal{W}^\mu, <^\mu)$), each of which satisfies (i)–(iii) in the hypothesis with respect to the common domain of the functions.

Clearly, for all $\mu \in \mathbf{N}^n$, $n \in \mathbf{N}$; $W \in \mathcal{W}$; $i \in \mathbf{N}$;

$$\mathrm{Cl} \bigcup \{f^{\mu,i}(V^\mu) : V \ll W\} \cap f^{\mu,i,j}(W^{\mu,i}) = \varnothing, \quad \text{for all } j \in \mathbf{N};$$

and therefore

$$\mathrm{Cl} \bigcup \{f^{\mu,i}(V^\mu) : V \ll W\} \subset \bigcup \{f^{\mu,i,j}(V^{\mu,i}) : V \ll W, j \in \mathbf{N}\};$$

and we can define $g^{\mu,i}$, for all $\mu \in \mathbf{N}^n$, $n \in \mathbf{N}$, and all $i \in \mathbf{N}$, by the formula

$$g^{\mu,i}(W) = f^{\mu,i}(W^\mu), \quad \text{for all } W \in \mathcal{W},$$

and let

$$G \equiv \{g^{\mu,i} : \mu \in \mathbf{N}^n, n \in \mathbf{N}; i \in \mathbf{N}\}.$$

REMARKS. 1. In Theorem 3, we can evidently restrict out attention to those well ordered open covers that are order isomorphic to their own cardinalities.

2. The proof of Theorem 3 can easily be modified to give the following result: On every well ordered open cover $(\mathcal{U}, <)$ of every paracompact (Hausdorff) space $(X, \mathfrak{T})$, there are functions f^i, $i \in \mathbf{N}$, into $\mathfrak{T}$ such that for every $i \in \mathbf{N}$, $f^i[\mathcal{U}]$ covers X and for each $U \in \mathcal{U}$, $f^i(U) \subset U$ and

$$\mathrm{Cl} \bigcup \{f^i(V) : V < U\} \subset \bigcup \{f^{i+1}(V) : V < U\}.$$

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