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PEDERSEN IDEAL AND GROUP ALGEBRAS

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ABSTRACT. For a locally compact T_2 group G which has an open subgroup of polynomial growth (e.g., G a group that has a compact neighbourhood invariant under inner automorphisms or G a compact extension of a locally compact nilpotent group) the intersection of the Pedersen ideal of the group C^*-algebra with $L^1(G)$ is dense in $L^1(G)$ (Theorem 1). For groups with small invariant neighbourhoods this intersection is the smallest dense ideal of $L^1(G)$, and it consists exactly of those $f \in L^1(G)$ whose "Fourier transform" vanishes outside some (closed) quasicompact subset of $\hat{G}$ (Theorem 3); the Pedersen ideal of $C^*(G)$ is described as the set of all $a \in C^*(G)$ for which $\{\pi \in \hat{G}: \pi(a) \neq 0\}$ is contained in some (closed) quasicompact subset of $\hat{G}$ (Theorem 2).

1. Introduction. In [11] G. K. Pedersen proved that every C^*-algebra A has a smallest dense order-related ideal K_A, and in 1975 K. B. Laursen and A. M. Sinclair showed that K_A (the so-called Pedersen ideal of A) is the smallest ideal among all dense ideals of A [5]. In [12] Pedersen asked whether $K_G \cap L^1(G)$ is dense in $L^1(G)$ where G is a locally compact T_2 group, $L^1(G)$ its group algebra and K_G the Pedersen ideal of the group C^*-algebra $C^*(G)$. The answer is affirmative in the case where G is abelian or compact (well known) or a connected real nilpotent Lie group [13].

In this note we shall show that the answer is affirmative even in the case where G has at least one compact neighbourhood U of the group identity such that $\lambda(U^k) = O(k^n)$ for some fixed $n \in \mathbf{N}$ (λ left Haar-measure). Examples of such groups are, e.g., groups that contain an open subgroup which is a compact extension of a (locally compact) nilpotent group, and IN-groups ($G \in [\mathrm{IN}] \leftrightarrow G$ has a compact invariant neighbourhood); the latter because the open subgroup G_F consisting of all elements with relatively compact conjugacy classes has polynomial growth [10].

In the special case of a SIN-group G ($G \in [\mathrm{SIN}] \leftrightarrow G$ has a fundamental system of compact invariant (under inner automorphisms) neighbourhoods of the identity of G) we show that K_G consists exactly of those $a \in C^*(G)$ for which $\{\pi \in \hat{G}; \pi(a) \neq 0\}$ is contained in a quasicompact subset of $\hat{G}$, and that $L^1(G) \cap K_G$ is the smallest dense ideal of $L^1(G)$.

The question whether there is a locally compact group G at all for which $L^1(G) \cap K_G$ is not dense in $L^1(G)$ or even $L^1(G) \cap K_G = \{0\}$ still seems to be open.

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REMARK. The existence of a smallest dense ideal in $L^1(G)$, $G \in$ [SIN], has also been shown in [4].

2. Pedersen ideal and groups of polynomial growth. Let A be a C^*-algebra. The Pedersen ideal K_A can be obtained in the following way (see [12]): K_A is the complex linear span of the invariant face generated by the set

$$K_A^{+00} := \{x \in A^+ : \exists y \in A^+ \text{ with } x = xy\}.$$

(A face F is a convex cone in A^+ such that: $x \in F$, $z \in A^+$, $z \leq x \Rightarrow z \in F$; F is called invariant $\Leftrightarrow a^*Fa \subseteq F$ $\forall a \in A \Leftrightarrow u^*Fu = F$ $\forall u \in \tilde{A}$, u unitary, $\tilde{A}$ the C^*-algebra obtained by adjunction of a unit (if A does not have a unit)).

For group algebras of groups with polynomial growth, J. Dixmier has found in [1] a functional calculus which turns out to be a very useful tool in harmonic analysis (see e.g. [7] and [8]).

Let C_n ($n \in \mathbf{N}$) denote the set of all functions $\varphi: \mathbf{R} \to \mathbf{C}$, $\varphi(0) = 0$ which have continuous and integrable derivatives of order $\leq n + 3$. Let V be a compact neighbourhood of the identity of a locally compact group G such that $\lambda(V^k) = O(k^n)$, $f = f^* \in L^1(G) \cap L^2(G)$ such that $f = 0$ outside V. Then for every $\varphi \in C_n$ the integral

$$\varphi\{f\} := \frac{1}{2\pi} \int_{\mathbf{R}} \exp(i\lambda f) \hat{\varphi}(\lambda)\, d\lambda$$

converges in $L^1(G)$ ($\hat{\varphi}$ is the Fourier transform of φ; "exp" is with respect to convolution) and for every $*$-representation π of $L^1(G)$ on a Hilbert space

$$\pi(\varphi\{f\}) = \varphi(\pi(f)) \tag{$*$}$$

where the right side is defined by the usual operational calculus on the hermitian operator $\pi(f)$.

THEOREM 1. *Let G be a locally compact T_2 group with a compact neighbourhood V of the identity e of G such that $\lambda(V^k) = O(k^n)$ for some $n \in \mathbf{N}$ (equivalently: G contains an open subgroup of polynomial growth). Then $L^1(G) \cap K_G$ is a dense ideal in $L^1(G)$ (where K_G denotes the Pedersen ideal of $C^*(G)$).*

PROOF. Take a compact neighbourhood $U = U^{-1}$ of e such that $U^{2(n+4)} \subseteq V$, $(g_i)_{i \in I}$ a bounded approximate unit for $L^1(G)$, $g_i: G \to \mathbf{R}$ continuous, $g_i(x) > 0$ $\forall x \in G$, $\operatorname{supp}(g_i) \subseteq U$, $\|g_i\|_1 = 1$, and define $f_i := g_i^* * g_i$. Take $\varphi \in C_n$ such that

$$\varphi(t) = t^{n+4} \quad \text{for all } t \text{ with } |t| \leq 1 = \|f_i\|_1,$$
$$\varphi(t) \geq 0 \quad \forall t \geq 0.$$

Now fix i, choose $\varepsilon > 0$. There is a real valued $\psi_{i,\varepsilon} \in C_n$ with

$$\psi_{i,\varepsilon}(t) = \varphi(t) \quad \forall t \in \mathbf{R} \setminus [-1, +1],$$
$$|\psi_{i,\varepsilon}^{(\alpha)}(t) - \varphi^{(\alpha)}(t)| < \varepsilon/A_i \quad \forall t \in [-1, +1], \alpha = 0, 1, \ldots, n + 3,$$

and

$$\psi_{i,\varepsilon}(t) = 0 \quad \forall |t| < \delta_{i,\varepsilon} \quad \text{for some } \delta_{i,\varepsilon} \in (0, 1).$$

[1, Lemme 8] where A_i is the constant A (independent of ε) in the proof of [1, Théorème 1.b]. Without loss of generality $\psi_{i,\varepsilon}(t) \geq 0\ \forall t \geq 0$.

Now we have $\varphi\{f_i\} = f_i^{n+4}$ (exponent with respect to convolution) and

$$\|\psi_{i,\varepsilon}\{f_i\} - f_i^{n+4}\|_1 < \varepsilon$$

[1, Théorème 1.b], hence $\{\psi_{i,\varepsilon}\{f_i\}: i \in I, 0 < \varepsilon \leq 1\}$ is a bounded approximating set for $L^1(G)$. If we can show all $\psi_{i,\varepsilon}\{f_i\}$ to be in K_G the proof is finished, since $\{f * \psi_{i,\varepsilon}\{f_i\}: f \in L^1(G), i \in I, 0 < \varepsilon \leq 1\}$ is dense in $L^1(G)$.

By construction $f_i \in C^*(G)^+$, and its spectrum $\sigma_{C^*(G)}(f_i)$ is contained in [0, 1]. Since $\psi_{i,\varepsilon}(f_i) = \psi_{i,\varepsilon}\{f_i\}$ (take for π the universal representation in (*); $\psi_{i,\varepsilon}(f_i)$ means the usual functional calculus for C^*-algebras) and $\psi_{i,\varepsilon}(t) \geq 0\ \forall t \geq 0$, we have $\psi_{i,\varepsilon}(f_i) \in C^*(G)^+$. Choose a function $\rho_{i,\varepsilon}: \mathbf{R} \to [0, \infty)$, $\rho_{i,\varepsilon} \in C_n$ such that

$$\rho_{i,\varepsilon}(t) = 1 \quad \forall t \in [\delta_{i,\varepsilon}, 1]$$

and

$$\rho_{i,\varepsilon}(t) = 0 \quad \forall t \in \mathbf{R} \setminus [1/2 \cdot \delta_{i,\varepsilon}, 2].$$

Now $\rho_{i,\varepsilon}\{f_i\} \in L^1(G)$, $\rho_{i,\varepsilon}\{f_i\} = \rho_{i,\varepsilon}(f_i) \in C^*(G)^+$ and $\psi_{i,\varepsilon}(f_i)\rho_{i,\varepsilon}(f_i) = \psi_{i,\varepsilon}(f_i)$, hence $\psi_{i,\varepsilon}\{f_i\} \in K_G^{+00} \subseteq K_G$.

REMARK 1. The properties of the bounded approximating set in the proof of Theorem 1 show at once that $I \cap K_G$ is dense in I for every left (or right) ideal in $L^1(G)$ (I not necessarily closed).

REMARK 2. The construction of the Pedersen ideal K_A of a C^*-algebra A shows at once that for all elements $x \in K_A$ the set $\{\pi \in \hat{A}: \pi(x) \neq 0\}$ is contained in a quasicompact subset of $\hat{A}$ since $ab = b$ ($a, b \in A^+$) implies $\{\pi \in \hat{A}: \pi(b) \neq 0\} \subseteq \{\pi \in \hat{A}: \|\pi(a)\| \geq 1\}$. Hence we have the following

COROLLARY. *G as in Theorem 1. The set of all $f \in L^1(G)$ for which the "Fourier transform" $\hat{f}$, $\hat{f}(\pi) := \pi(f)$, $\pi \in \hat{G}$, vanishes outside a quasicompact set of $\hat{G}$ is dense in $L^1(G)$.*

3. Pedersen ideal and SIN-groups. For SIN-groups we get more detailed information than in the corollary above:

THEOREM 2. *Let $G \in [SIN]$, K_G the Pedersen ideal of $C^*(G)$, $J_G := \{a \in C^*(G);$ $\hat{a}$ vanishes outside a quasicompact subset of $\hat{G}\}$ (where $\hat{a}(\pi) := \pi(a)\ \forall \pi \in \hat{G}$). Then $J_G = K_G$.*

PROOF. We only have to show $J_G \subseteq K_G$. Consider the following mappings t and p:

$$t: \hat{G} \to \operatorname{Prim} C^*(G), \qquad \pi \mapsto \ker \pi,$$
$$p: \operatorname{Prim} C^*(G) \to G\text{-Max } C^*(G_F) \cong E(G_F, G), \qquad P \mapsto P \cap C^*(G_F),$$

where G-Max $C^*(G_F)$ denotes the ideals of $C^*(G_F)$ which are maximal among the G-invariant modular ideals; G-Max $C^*(G_F)$ with hull-kernel topology is homeomorphic to $E(G_F, G)$ (the extreme points of the set of all G-invariant continuous positive definite functions γ on G_F with $\gamma(e) = 1$) with the topology of compact convergence; the homeomorphism

$$E(G_F, G) \to G\text{-Max } C^*(G_F) \text{ is given by}$$
$$\gamma \mapsto \{a \in C^*(G_F): \langle a^*a, \gamma\rangle = 0\} \qquad [9, (4)].$$

For each $P \in \text{Prim } C^*(G)$ there is a continuous positive definite indecomposable function φ, $\varphi(e) = 1$ with $P = \ker \pi_\varphi$, and $P \cap C^*(G_F)$ corresponds to $(\varphi|G_F)^G \in E(G_F, G)$, which is defined by

$$(\varphi|G_F)^G(n) := \int_{\overline{I(G_F,G)}} \varphi(\beta^{-1}(n))\, d\beta$$

where $\overline{I(G_F, G)}$ is a compact group: the closure of the restrictions to G_F of the inner automorphisms of G. The mapping p: $\ker \pi_\varphi \mapsto (\varphi|G_F)^G$ is well-defined from $\text{Prim } C^*(G)$ onto $E(G_F, G)$ (even continuous and proper). See [2].

Now take an arbitrary $a \in J_G$, choose $L \subseteq \text{Prim } C^*(G)$ quasicompact with $t^{-1}(L) \supseteq \{\pi \in \hat{G}: \pi(a) \neq 0\}$. Since the "Fourier transform" of the hermitian and positive parts of a vanish outside $t^{-1}(L)$ too, $a \geqslant 0$ without loss of generality. Since the algebra of functions in $L^1(G_F)$ that are central in $L^1(G)$ is a completely regular Banach algebra with maximal ideal space $E(G_F, G)$ (see [3, (4)] or [6, (2.4)]) we can get $f \in L^1(G_F)$, central in $L^1(G)$, with

$$\hat{f}(\alpha) := \int_{G_F} f(x)\alpha(x)\, dx \geqslant 0 \quad \forall \alpha \in E(G_F, G),$$
$$\hat{f}(\alpha) = 1 \quad \forall \alpha \in p(L) \subseteq E(G_F, G).$$

Then $f \in C^*(G_F)^+ \subseteq C^*(G)^+$ and $\pi(f) = \mathrm{id}_{H_\pi}$ $\forall \pi \in t^{-1}(L) \subseteq \hat{G}$, hence $fa = a$, hence $a \in K_G^{+\infty} \subseteq K_G$.

Let us check $\pi(f) = \mathrm{id}_{H_\pi}$ $\forall \pi \in t^{-1}(L)$. Let $\pi \in \hat{G}$, φ with $\pi_\varphi = \pi$:

$$\begin{aligned}\hat{f}\big((\varphi|G_F)^G\big) &= \int_{G_F} f(x)(\varphi|G_F)^G(x)\, dx = \int_{G_F} f^G(x)\varphi(x)\, dx \\ &= \int_G f(x)\varphi(x)\, dx = \int_G f(x)\big(\pi(x)\xi_\varphi, \xi_\varphi\big)\, dx \\ &= \big(\pi(f)\xi_\varphi, \xi_\varphi\big).\end{aligned}$$

Since f is central and π irreducible, $\pi(f)$ is a multiple of id_{H_π}, so we have

$$\pi(f) = \hat{f}\big((\varphi|G_F)^G\big) \cdot \mathrm{id}_{H_\pi} \quad \forall \pi \in \hat{G}, \varphi \text{ with } \pi_\varphi = \pi;$$

hence the assertion.

REMARK. In a SIN-group G each quasicompact subset of $\hat{G}$ is contained in a closed quasicompact subset of $\hat{G}$ (because the mapping p in the proof of Theorem 2 is continuous and proper).

LEMMA. $G \in [\text{SIN}]$, *I a dense ideal in* $L^1(G)$. *Then for every quasicompact set* $L \subseteq \text{Prim } C^*(G)$ *there is a* $u \in I$ *such that u is a unit for* $L^1(G)$ *modulo* $k(L) \cap L^1(G)$. $[k(L) := \bigcap\{P \in \text{Prim } C^*(G): P \in L\}]$.

PROOF. For L quasicompact take f as in the proof of Theorem 2; then f is a unit for $L^1(G)/(k(L) \cap L^1(G))$. Since $k(L) \cap L^1(G)$ is a modular ideal in $L^1(G)$, $I + (k(L) \cap L^1(G)) = L^1(G)$ [**14**, 2.6.8] and thus there is a $d \in k(L) \cap L^1(G)$ and $u \in I$ with $d + u = f$, hence $u - f \in k(L) \cap L^1(G)$ and thus u is a unit for $L^1(G)$ modulo $(k(L) \cap L^1(G))$.

THEOREM 3. *Let $G \in [SIN]$. Then there is a smallest dense ideal J in $L^1(G)$. J coincides with the intersection of $L^1(G)$ and the Pedersen ideal K_G of $C^*(G)$ and also with the set of all $h \in L^1(G)$ for which the "Fourier transform" $\hat{h}$ vanishes outside a quasicompact subset of $\hat{G}$.*

PROOF. $J := L^1(G) \cap K_G$ is dense in $L^1(G)$. Now let I be an arbitrary dense ideal in $L^1(G)$. For every $a \in J$ there exists a quasicompact set $L \subseteq \operatorname{Prim} C^*(G)$ with $L \supseteq \{P \in \operatorname{Prim} C^*(G);\ a \notin P\}$, hence by the lemma above there is a $u \in I$ with $ua + P = a + P\ \forall P \in L$, and for all $P \in \operatorname{Prim} C^*(G) \setminus L$ too (since then $a \in P$). Thus we have $ua = a$, hence $a \in I$.

The last assertion follows from Theorem 2.

4. Added in proof. 1. Let $G = G_{4,9}(0)$ be the group of all 3×3 matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & e^r & y \\ 0 & 0 & 1 \end{bmatrix}, \qquad r, x, y, z \in \mathbf{R}.$$

Then the intersection of all dense two-sided ideal in $L^1(G)$ is trivial. The intersection of the Pedersen ideal of $C^*(G)$ with $L^1(G)$ is trivial, too. (Communicated by Viktor Losert.)

2. Let G be a locally compact T_2 group. If $G_0 \in [\mathrm{IN}]$ (G_0 the identity component) or if G is a group of polynomial growth with symmetric group algebra $L^1(G)$, then there does exist a smallest dense two-sided ideal of $L^1(G)$ (V. Losert, resp., J. Ludwig).

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