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CHARACTERIZATION OF (r, s)-ADJACENCY GRAPHS OF COMPLEXES

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ABSTRACT. The (r, s)-adjacency graph of a simplicial complex K has been defined as the graph whose nodes are the r-cells of K with adjacency whenever there is incidence with a common s-cell. The (r, s)-adjacency graphs for $r > s$ have been characterized by graph coverings by Dewdney and Harary generalizing the result of Krausz for line-graphs ($r = 1$, $s = 0$). We now complete the characterization by handling the case $r < s$.

1. Introduction. Let S be a collection of distinct subsets called *simplexes* of a nonempty finite set V whose elements are called *nodes*. Then $K = (V, S)$ is a (*simplicial*) *complex* if it satisfies the condition that every nonempty subset of a simplex $x \in S$ is also a simplex.

The *dimension of a simplex* x in K is $r = |x| - 1$ and x is called an *r-simplex* or sometimes an *r-cell*. The *dimension of a complex* K is the maximum dimension of a simplex in it. A complex of dimension r is called an *r-complex*. Thus a 1-complex is a graph which has at least one line. If the graph is totally disconnected, it is, of course, a 0-complex. A *pure r-complex* is one in which every maximal simplex has dimension r.

Every complex K has an associated hypergraph whose edges are its maximal simplexes. Conversely given any hypergraph, we can construct its complex by including every nonempty subset of an edge as a simplex. Thus an r-complex is known as an hereditary rank-r hypergraph.

The *(r, s)-adjacency graph*, $r \neq s$, of a complex K denoted by $L_{rs}(K)$, in analogy with the standard notation for the line-graph $L(G)$, is the graph whose nodes are the r-simplexes of K, with two of these nodes adjacent whenever their r-simplexes are incident with a common s-simplex. Thus if K is a 1-complex, then $L_{10}(K)$ is its line-graph.

This concept was first suggested by Grünbaum [6] for $s = r - 1$ and has been investigated for $r > s$ by Dewdney and Harary [3], by Bermond, Sotteau, Heydemann, Germa in a series [1], [2], [8], and for $s = 0$ and $s = r - 1$ by Gardner [4], [5] and others.

Let $x_1, \ldots, x_t$ be simplexes of K, with no x_i contained in x_j for any $i \neq j$. Their *induced complex* is the subcomplex K' whose maximal simplexes are the x_i.

2. Structural characterization of adjacency graphs. Krausz [9] characterized line-graphs of 1-complexes by a suitable partition of the edges into complete subgraphs.

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In this statement, condition (i) is redundant because of the partition requirement, but it is useful for later generalization.

THEOREM A (KRAUSZ). *The graph G is a $(1, 0)$-adjacency graph if and only if the edges can be partitioned into a family of complete subgraphs G_i satisfying*:

(i) *Each edge is in exactly one G_i.*

(ii) *Each vertex is in no more than two G_i.*

Grünbaum [6] conjectured that this characterization could be extended to graphs which are $(r, r-1)$-adjacency graphs. Necessary conditions which are not sufficient are given in [2]. When $r > s$, necessary and sufficient conditions for (r, s)-adjacency graphs were developed in [3].

THEOREM B (DEWDNEY AND HARARY). *The graph G is an (r, s)-adjacency graph with $r > s$ if and only if there is a family G_i of subgraphs of G satisfying the following three conditions.*

(i) *Each edge lies in at most r and at least $s+1$ of the graphs G_i.*

(ii) *Each vertex lies in at most $r+1$ of the graphs G_i.*

(iii) *The intersection of any $s+1$ of the graphs G_i is either empty or a complete graph.*

The proof, in analogy with that of Krausz for line-graphs, constructs a complex whose nodes (0-simplexes) are the G_i and which has one r-simplex for each node v of G. The r-simplex corresponding to v contains the 0-simplex G_i whenever node v of G is in subgraph G_i.

A few comments are in order. The complete bipartite graph $G = K(2, \binom{2r}{r})$ is the $(2r-1, r-1)$-adjacency graph of some complex K. Such a complex K can be constructed by taking $4r$ nodes, partitioning them into two sets x and y of size $2r$ each and defining the $(2r-1)$-simplexes to be x, y and all sets consisting of r nodes from each of x and y. It is not possible however to cover the edges of G with a set of complete graphs which satisfies condition (i) of Theorem B. Therefore we cannot expect a characterization of the Krausz type which contains among the G_i a subset consisting of complete graphs which covers the edges of G.

Bermond, Heydemann and Sotteau [1] defined the *s-line-graph* of a hypergraph H, denoted $L_s(H)$, as the graph whose nodes are the edges of H with two edges adjacent if their intersection contains at least s nodes. Theorem B characterizes the $(s+1)$-line-graphs of rank-$(r+1)$ hypergraphs.

In an *exact (r, s)-adjacency graph*, two adjacent simplexes have a common s-simplex but not a common $(s+1)$-simplex. If (i) is replaced by

(i′) Each edge lies in exactly $s+1$ of the graphs G_i, then Theorem B characterizes the exact (r, s)-adjacency graphs.

We now finish the characterization of all (r, s)-adjacency graphs by deriving conditions for $r < s$.

THEOREM 1. *A graph G is the (r, s)-adjacency graph of a simplicial complex with $r < s$ if and only if the edges of G are covered by a set of complete subgraphs G_i of* *order* $\binom{s+1}{r+1}$ *which are grouped into subsets* $S_1, \ldots, S_p$ *such that the* G_i *and* S_j *satisfy the following conditions.*

(i) *The intersection of every subset of the* G_i*'s has the form* K_b *where* b *is a binomial coefficient of the form* $\binom{n+1}{r+1}$ *for some* $n \geq r$. *In this case there are exactly* $n + 1$ *sets* S_j *containing this subset of* G_i*'s.*

(ii) *Each* G_i *is in at most* $s + 1$ *sets* S_j.

PROOF. Let G be any graph with isolated nodes $u_1, \ldots, u_n$. Represent the u_j by node-disjoint r-simplexes, one for each u_j. Then G is an (r, s)-adjacency graph if and only if $G - \{u_1, \ldots, u_n\}$ is. Assume without loss of generality that G is connected.

We will first show that the conditions of the theorem are sufficient. We do this by constructing a complex K with a maximal simplex x_i for each graph G_i. For each G_i define disjoint sets T_i of cardinality $s + 1 - |\{S_j: G_i \in S_j\}|$. Set $x_i = \{S_j: G_i \in S_j\} \cup T_i$. Let K be the complex induced by the maximal simplexes x_i.

We prove that $L_{rs}(K) \cong G$ by induction on $t - m$, where t is the number of G_i's. For the induction hypothesis, suppose that there is a one-to-one map from the nodes that lie in at least $m + 1$ G_i's onto the r-simplexes of K incident with at least $m + 1$ maximal simplexes, that is, s-simplexes. In addition suppose that the map has the property that a node u is mapped to an r-simplex x, with $x = \{S_1, \ldots, S_{r+1}\}$ or $x = \{S_1, \ldots, S_j, t_{j+1}, \ldots, t_{r+1}\}$ with $t_{j+1}, \ldots, t_r \in T_i$ for some i, only if $S_1 \cap \cdots \cap S_{r+1} = \{G_i: u \in G_i\}$.

Let $m = 0$. The intersection of all the G_i's is K_b with $b = \binom{n+1}{r+1}$. By condition (i), there are exactly $n + 1$ S_j's containing all the graphs G_i and there are therefore exactly $\binom{n+1}{r+1}$ r-simplexes lying in the intersection of all the x_i's. Fix any one-to-one map between the nodes of the K_b and these r-simplexes.

Now let $m > 0$ be given. Fix an m-set $\{G_{i_1}, \ldots, G_{i_m}\}$ with $H = G_{i_1} \cap \cdots \cap G_{i_m} = K_b$, $b = \binom{n+1}{r+1}$. Using condition (i) again, there are exactly $n + 1$ S_j's containing this subset of G_i's and therefore exactly $\binom{n+1}{r+1}$ r-simplexes in $x_{i_1} \cap \cdots \cap x_{i_m} = \{S_{j_1}, \ldots, S_{j_{n+1}}\}$. Some of these r-simplexes are already the image of a node of G under the map defined by the induction hypothesis. The number of such r-simplexes is equal to the number of nodes in the union of the graphs $H \cap G_{i_{m+1}}$ taken over all $i_{m+1} \neq i_1, \ldots, i_m$. As this is just the number of r-simplexes in the corresponding union of the $x_{i_1} \cap \cdots \cap x_{i_m} \cap x_{i_{m+1}}$ the map can be extended in a one-to-one fashion to map the nodes of $H = G_{i_1} \cap \cdots \cap G_{i_m}$ to the r-simplexes of $x_{i_1} \cap \cdots \cap x_{i_m}$.

Observe that no unassigned node is in two distinct intersections of m graphs G_i. Thus the map will be well-defined if it is extended in this way and will cover all nodes in any intersection of m graphs G_i. The map is one-to-one by definition and onto by construction.

Finally we verify that two r-simplexes are the image of adjacent nodes of G if and only if the simplexes are incident with a common s-simplex. Let x and y be two r-simplexes whose preimages in G are u and v. If u and v are adjacent, then the edge uv is in some G_i and x and y are both incident with the s-simplex x_i.

Conversely, if x and y are incident with an s-simplex x_i then their preimages must be contained in G_i.

To prove the necessity of the conditions, suppose that G is the (r, s)-adjacency graph of a complex K. Let x_i be an s-simplex of K. Then there are $\binom{s+1}{r+1}$ r-simplexes in x_i and the image of x_i in G is

$$K\binom{s+1}{v+1} = G_i.$$

If m distinct r-simplexes overlap on $n + 1$ points, their images in G will be a complete graph on $\binom{n+1}{s+1}$ nodes.

For each 0-simplex j of K, define the set $S_j = \{G_i\colon\ x_i$ is an s-simplex with $j \in x_i\}$. Clearly the S_j satisfy conditions (i) and (ii). $\square$

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