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THE RADON TRANSFORM ON A FAMILY OF CURVES IN THE PLANE[1]

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ABSTRACT. Inversion formulas are given for Radon's problem when the line integrals are evaluated along curves given, for a fixed (p, ϕ), by $r^{\alpha} \cos|\alpha(\theta - \phi)| = p^{\alpha}$, where α is real, $\alpha \neq 0$.

Radon's problem, the recovery of a function from its integrals along straight lines in the plane, has received considerable attention in recent years because of its many practical applications, notably in medicine. The reason that straight lines are significant is that the probing agents in many applications, photons, charged particles, and phonons, obey Newton's first law and travel in straight lines unless they undergo interactions. It is easy to conceive of situations in which the probing agents do not follow straight lines, so Radon's problem needs to be generalized to other curves. Such generalizations have been made for certain ellipses and circles **[5]**, [7], and for circles through the origin [1], [3]. When certain Fourier expansions are made, the treatment of these last circles is so similar to the treatment of straight lines using the same expansions [1], [2], [6] that it seemed desirable to see whether they could both be special cases of a more general set of curves in the plane. The purpose of this note is to treat a family of curves in the plane which does contain these as special cases.

Let (r, θ) and (p, ϕ) be polar coordinates in the plane and consider the curves given by

$$r^{\alpha} \cos\{\alpha(\theta - \phi)\} = p^{\alpha}, \qquad |\theta - \phi| \leqslant \pi/2\alpha. \tag{1a}$$

We could let α be a real number not equal to zero, but it is convenient to treat positive and negative values of α separately. Hence in (1a) we restrict α to $\alpha > 0$, and we shall refer to these curves as "α-curves". For $\alpha < 0$ we write $\beta = -\alpha$ and refer to the curves

$$p^{\beta} \cos\{\beta(\theta - \phi)\} = r^{\beta}, \qquad |\theta - \phi| \leqslant \pi/2\beta \tag{1b}$$

as "β-curves". For given values of α or β and (p, ϕ), (1a) and (1b) describe curves which are symmetrical about the line $\theta = \phi$. The α-curves tend to infinity as $|\theta - \phi| \to \pi/2\alpha$; they intersect themselves at least once if $0 < \alpha < 1/2$, but they do not intersect themselves if $\alpha \geqslant 1/2$. Well-known special cases are parabolae,

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straight lines, and one-branched hyperbolae for $\alpha = 1/2$, 1, and 2 respectively. The β-curves tend to the origin as $|\theta - \phi| \to \pi/2\beta$; they intersect themselves at least once if $0 < \beta < 1/2$ but they do not intersect themselves if $\beta \geqslant 1/2$. Well-known special cases are cardioids, circles through the origin, and one-branched lemniscates of Bernoulli for $\beta = 1/2$, 1, and 2 respectively. Many α- and β-curves occur as the orbits of particles moving under a central force, or the orbits of charged particles moving in certain magnetic fields. The inversion of an α-curve in the unit circle $((r, \theta) \to (1/r, \theta))$ is a β-curve with $\beta = \alpha$, and conversely. A related result is that the Radon transform of a delta-function taken with respect to an α-curve is concentrated on a β-curve with $\beta = \alpha$, and conversely.

Let $f(r, \theta)$ be a smooth rapidly decreasing function and let $\hat{f}(p, \phi)$ be the Radon transform of f along some α-curve specified by (p, ϕ). Then we can write

$$\hat{f}(p, \phi) = \int_\alpha f(r, \theta)\, ds_\alpha, \tag{2}$$

where ds_α is an element of length along the α-curve. Using (1a), (2) becomes

$$\hat{f}(p, \phi) = p\int_{-\pi/2\alpha}^{\pi/2\alpha} f\left(\frac{p}{\cos^{1/\alpha}(\alpha\psi)}, \phi + \psi\right)\frac{d\psi}{\cos^{1+1/\alpha}(\alpha\psi)}. \tag{3}$$

Now write

$$f(r, \theta) = \sum_{l=-\infty}^{+\infty} f_l(r)e^{il\theta}, \tag{4}$$

where

$$f_l(r) = (1/2\pi)\int_0^{2\pi} f(r, \theta)e^{-il\theta}\, d\theta. \tag{5}$$

Then it is known that $f_l(r)$ is smooth and rapidly decreasing and also that $f_l(r) = r^l g(r)$ where g is an even, smooth, rapidly decreasing function of r. Substituting (4) into (3) we have

$$\hat{f}(p, \phi) = p\sum_{l=-\infty}^{+\infty} e^{il\phi}\int_{-\pi/2\alpha}^{\pi/2\alpha} f_l\left(\frac{p}{\cos^{1/\alpha}(\alpha\psi)}\right)\frac{e^{il\psi}\, d\psi}{\cos^{1+1/\alpha}(\alpha\psi)}. \tag{6}$$

Now if $\hat{f}$ is expanded in a Fourier series:

$$\hat{f}(p, \phi) = \sum_{l=-\infty}^{+\infty} \hat{f}_l(p)e^{il\phi}, \tag{7}$$

then

$$\hat{f}_l(p) = p\int_{-\pi/2\alpha}^{\pi/2\alpha} f_l\left(\frac{p}{\cos^{1/\alpha}(\alpha\psi)}\right)\frac{d\psi}{\cos^{1+1/\alpha}(\alpha\psi)}. \tag{8}$$

Putting $r = p/\cos^{1/\alpha}(\alpha\psi)$, (8) becomes

$$\hat{f}_l(p) = 2\int_p^\infty f_l(r)\frac{\cos\{(l/\alpha)\cos^{-1}(p/r)^\alpha\}}{(1 - (p/r)^{2\alpha})^{1/2}}\, dr. \tag{9a}$$

The Radon transform on the β-curves can be treated in the same way and, using (4) and (7) for the expansion of f and $\hat{f}$, one finds that

$$\hat{f}_l(p) = 2\int_0^p f_l(r)\frac{\cos\{(l/\beta)\cos^{-1}(r/p)^{\beta}\}}{(1-(r/p)^{2\beta})^{1/2}}\,dr. \tag{9b}$$

(9a) and (9b) may be simplified by making the substitutions r^{α} or $r^{\beta} = s$, p^{α} or $p^{\beta} = q$ and defining

$$F_l(s) = (1/\alpha)f_l(s^{1/\alpha})s^{1/\alpha - 1}, \tag{10}$$

$$\hat{F}_l(q) = \hat{f}(q^{1/\alpha}) \tag{11}$$

for the α-curves, and defining F_l and $\hat{F}_l$ for the β-curves by substituting β for α in (10) and (11). When this is done we have

$$\hat{F}_l(q) = 2\int_q^{\infty} F_l(s)\frac{\cos\{(l/\alpha)\cos^{-1}(q/s)\}}{(1-(q/s)^2)^{1/2}}\,ds \tag{12a}$$

for the α-curves, and

$$\hat{F}_l(q) = 2\int_0^{q} F_l(s)\frac{\cos\{(l/\beta)\cos^{-1}(s/q)\}}{(1-(s/q)^2)^{1/2}}\,ds \tag{12b}$$

for the β-curves. (12a) and (12b) are integral equations for the F_l given the $\hat{F}_l$, and if they can be solved we have inversion formulae for Radon's problem on this family of curves.

The solution of (12a) and (12b) follows the same pattern as in **[1]** for the cases $\alpha = \beta = 1$. Multiply (12a) by

$$\frac{\cosh\{(l/\alpha)\cosh^{-1}(q/t)\}}{q((q/t)^2 - 1)^{1/2}}$$

and integrate over q:

$$\begin{aligned}&\int_t^{\infty}\frac{\hat{F}_l(q)\cosh\{(l/\alpha)\cosh^{-1}(q/t)\}}{((q/t)^2-1)^{1/2}}\,\frac{dq}{q}\\ &\quad = 2\int_t^{\infty}\frac{\cosh\{(l/\alpha)\cosh^{-1}(q/t)\}}{((q/t)^2-1)^{1/2}}\,\frac{dq}{q}\int_q^{\infty}\frac{F_l(s)\cos\{(l/\alpha)\cos^{-1}(q/s)\}}{(1-(q/s)^2)^{1/2}}\,ds.\end{aligned} \tag{13}$$

Because F_l is rapidly decreasing the order of integration on the right-hand side may be changed and the right-hand side becomes

$$2\int_t^{\infty}F_l(s)\,ds\int_t^{s}\frac{\cosh\{(l/\alpha)\cosh^{-1}(q/t)\}\cos\{(l/\alpha)\cos^{-1}(q/s)\}}{(1-(q/s)^2)^{1/2}((q/t)^2-1)^{1/2}}\,\frac{dq}{q}. \tag{14}$$

But it is shown in the Appendix that the q-integral in (14) is just $\pi/2$, hence (13) becomes

$$\pi\int_t^{\infty}F_l(s)\,ds = \int_t^{\infty}\hat{F}_l(q)\frac{\cosh\{(l/\alpha)\cosh^{-1}(q/t)\}}{((q/t)^2-1)^{1/2}}\,\frac{dq}{q}, \tag{15a}$$

and differentiation of this yields

$$(16a)\qquad F_l(t) = -\frac{1}{\pi}\frac{d}{dt}\int_t^\infty \frac{\hat{F}_l \cosh\{(l/\alpha)\cosh^{-1}(q/t)\}}{((q/t)^2-1)^{1/2}}\frac{dq}{q}$$

as the solution of (12a). The derivative may be taken under the integral sign to give the alternative form

$$(17a)\qquad F_l(t) = -\frac{1}{\pi}\int_t^\infty \frac{d\hat{F}_l(q)}{dq}\cdot\frac{\cosh\{(l/\alpha)\cosh^{-1}(q/t)\}\,dq}{(q^2-t^2)^{1/2}}$$

(12b) may be treated in the same way by multiplying it by

$$\frac{\cosh\{(l/\beta)\cosh^{-1}(t/q)\}}{q((t/q)^2-1)^{1/2}}$$

and integrating q from 0 to t. The order of integration on the right-hand side may again be changed, and use of (A8) instead of (A7) yields

$$(15b)\qquad \pi\int_0^t F_l(s)\,ds = \int_0^t \hat{F}_l(q)\frac{\cosh\{(l/\beta)\cosh^{-1}(t/q)\}}{((t/q)^2-1)^{1/2}}\frac{dq}{q},$$

from which one obtains

$$(16b)\qquad F_l(t) = \frac{1}{\pi}\frac{d}{dt}\int_0^t \hat{F}_l(q)\frac{\cosh\{(l/\beta)\cosh^{-1}(t/q)\}}{((t/q)^2-1)^{1/2}}\frac{dq}{q},$$

and

$$(17b)\qquad F_l(t) = \frac{1}{\pi t}\int_0^t \frac{d\hat{F}_l(q)}{dq}\,\frac{\cosh\{(l/\beta)\cosh^{-1}(t/q)\}\,q\,dq}{(t^2-q^2)^{1/2}}.$$

(16) and (17) provide the inversion formulae for Radon's problem for this family of curves. They also demonstrate the "hole" theorems appropriate to the two cases: for the α-curves it is only necessary to know $\hat{f}(p, \phi)$ for $p > r_0$ in order to determine $f(r_0, \theta)$; for the β-curves it is only necessary to know $\hat{f}(p, \phi)$ for $p < r_0$ in order to determine $f(r_0, \theta)$.

For a fixed t the cosh term in (16a) and (17a) becomes large as $q \to \infty$ in a way which increases rapidly with increasing l. Hence in a practical problem in which $\hat{F}_l$ is known only from noisy data the noise will be propagated badly into the calculation of $\hat{F}_l$ for small t unless l is small. Likewise, in equations (16b) and (17b) noise will be propagated badly into the calculation of F_l from noisy data for $\hat{F}_l$ near the origin unless l is small.

When α or $\beta = 1$ in (9), (6) and (17) the cos and cosh terms can be written as Tschebycheff polynomials of the first kind since, if $n = 0, 1, 2, 3, \ldots,$ $T_n(x) = \cos\{n \cos^{-1} x\}$ if $|x| \leq 1$, and $T_n(x) = \cosh\{n \cosh^{-1} x\}$ if $|x| > 1$. In these cases one obtains some nice properties of the f_l and $\hat{f}_l$ such as: (i) the number of zeros $\hat{f}_l$ must have, (ii) the relation between the Hankel transform of f_l and the Fourier transform of $\hat{f}_l$ for $\alpha = 1$, and, as a consequence of (ii), (iii) relations between orthogonal expansions of f_l and $\hat{f}_l$. The F_l and $\hat{F}_l$ appear to have similar properties only for the special cases α or $\beta = 1/m$, $m = 1, 2, 3, \ldots$. They will not be given here because they are simple extensions of results given in [1], [2], [3], and [6].

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Appendix: Evaluation of two integrals. Consider the integral

$$I = \int_t^s \frac{\cos\{\alpha \cos^{-1}(q/s)\}\cosh\{\alpha \cosh^{-1}(q/t)\}}{(1-(q/s)^2)^{1/2}((q/t)^2-1)^{1/2}} \frac{dq}{q} \tag{A1}$$

where α is real and $0 < t < s$. For $\alpha = 0, 1, 2, \ldots$ this is the integral used in reference A, and its value was given as $\pi/2$. It is readily seen that

$$I = -\frac{s^2t^2}{\alpha^2}\frac{d^2}{ds\,dt}\int_t^s \sin\{\alpha \cos^{-1}(q/s)\}\sinh\{\alpha \cosh^{-1}(q/t)\}\frac{dq}{q^3}. \tag{A2}$$

Let $F(a, b; c; x)$ be the ordinary hypergeometric function defined by

$$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!} \tag{A3}$$

then the sin and sinh functions in (A2) can be written as hypergeometric functions, and (A2) becomes

$$\begin{aligned} I = -s^2t^2\frac{d^2}{ds\,dt}\int_t^s &(1-(q/s)^2)^{1/2}((q/t)^2-1)^{1/2}F(a, b; 3/2; 1-(q/s)^2) \\ &\cdot F(a, b; 3/2; 1-(q/t)^2)\frac{dq}{q^3} \end{aligned} \tag{A4}$$

where $a = (1+\alpha)/2$ and $b = (1-\alpha)/2$.

If the hypergeometric functions in (A4) are written as their defining series, (A4) becomes a double sum over n and m of integrals of the form

$$\int_t^s (1-(q/s)^2)^{n+1/2}((q/t)^2-1)^{m+1/2}\frac{dq}{q^3}.$$

Differentiating these with respect to s and t as in (A4) they become integrals which are just beta-functions. One of the sums in I is then itself a hypergeometric function and (A4) becomes

$$I = \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(1)_k}\frac{(1-(t/s)^2)^k}{k!}F(a, b; k+1; 1-(s/t)^2). \tag{A5}$$

Now Hansen [4, 65.2.4, p. 426] gives the following result.

$$\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(d)_k}\frac{x^k}{k!}F\left(d-b, c; k+d; \frac{x}{x-1}\right) = (1-x)^c F(b, a+c; d; x).$$

Putting $1 - t^2/s^2 = x$, $d = 1$, $a = (1+\alpha)/2$, $b = (1-\alpha)/2$, (A5) becomes

$$I = \frac{\pi}{2}(1-x)^{(1-\alpha)/2}F\left(\frac{1-\alpha}{2}, 1; 1; x\right), \tag{A6}$$

and, since

$$F\left(\frac{1-\alpha}{2}, 1; 1; x\right) = (1-x)^{-((1-\alpha)/2)},$$

(A6) becomes

(A7) $$I = \pi/2.$$

To deal with the β-curves, making the substitution $q = st/x$ in (A1) yields

(A8) $$\int_t^s \frac{\cos\{\beta\cos^{-1}(t/x)\}\cosh\{\beta\cosh^{-1}(s/x)\}}{(1-(t/x)^2)^{1/2}((s/x)^2-1)^{1/2}}\,\frac{dx}{x} = \frac{\pi}{2}.$$

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