PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 2, October 1981

COMPOSING FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. We find necessary and sufficient conditions on g such that $f \circ g$ (resp. $g \circ f$) is a function of bounded variation for all f of bounded variation.

Let $\mathbf{I} = [0, 1]$, and let $\mathbf{BV}$ denote the set of functions f: $\mathbf{I} \to \mathbf{I}$ of bounded variation. $\mathbf{BV}$ is not closed under composition (example: $f \circ g$ where $f(x) = \sqrt{x}$; $g(0) = 0$, $g(x) = x^2 \sin^2(1/x)$ for $x \neq 0$). Here we characterize those functions g whose right (resp. left) composition operation $f \to f \circ g$ (resp. $f \to g \circ f$) preserves $\mathbf{BV}$.

If f: $\mathbf{I} \to \mathbf{R}$, define $v(f, a, b)$ as the variation of f on the subinterval $[a, b]$, and $v(f) = v(f, 0, 1)$. For a positive integer N, let $J_N = \{X \subseteq I$: X can be expressed as a union of N intervals$\}$ (where the intervals may be open or closed at either end and we allow singletons as degenerate closed intervals). Since any interval is a union of two subintervals, we see $J_N \subset J_{N+1}$. A function f: $\mathbf{I} \to \mathbf{R}$ is said to be of *N-bounded variation* if $f^{-1}([a, b]) \in J_N$ for all $[a, b] \subseteq \mathbf{R}$. Let $\mathbf{BV}(N)$ be the set of all functions f: $\mathbf{I} \to \mathbf{I}$ of N-bounded variation, and $\mathbf{BV}'(N)$ the set of all bounded functions f: $\mathbf{I} \to \mathbf{R}$ of N-bounded variation.

LEMMA 1. *Every function in $\mathbf{BV}'(N)$ is of bounded variation.*

PROOF. Suppose $h \in \mathbf{BV}'(N)$ and $|h(x)| \leqslant M$ for all $x \in \mathbf{I}$. We show $v(h) \leqslant 4M(N + 1)$. If $v(h) > 4M(N + 1)$, then there exists a partition $\{x_0, \ldots, x_n\}$ of $\mathbf{I}$ with

$$\sum_{i=1}^{n} |h(x_i) - h(x_{i-1})| > 4M(N + 1).$$

Some $[a', b'] \subseteq [-M, M]$ with $a' < b'$ is covered more than $2(N + 1)$ times by intervals $[h(x_{i-1}), h(x_i)]$ or $[h(x_i), h(x_{i-1})]$. At least $N + 1$ are of the type $[h(x_{i-1}), h(x_i)]$, say, so

$$[a', b'] \subseteq \bigcap_{j=1}^{N+1} [h(y_j), h(z_j)]$$

where each $(y_j, z_j) = (x_{i-1}, x_i)$ for some i, and $y_1 < z_1 < \cdots < y_{N+1} < z_{N+1}$. Then $h(y_j) \leqslant a' < b' \leqslant h(z_j)$ for all j. Take a between a' and b'; take b greater than

Received by the editors March 1, 1980 and, in revised form, October 20, 1980.
1980 *Mathematics Subject Classification*. Primary 26A45; Secondary 04A05, 26A16.
Key words and phrases. Function of bounded variation, composition, Lipschitz condition.

0002-9939/81/0000-0478/$01.75

every $h(z_j)$. Then $h(y_j) \notin [a, b]$, $h(z_j) \in [a, b]$ for all j. Hence $h^{-1}([a, b])$ is union of no fewer than $N + 1$ intervals, and so is not in J_N, a contradiction. $\therefore h$ is of bounded variation.

LEMMA 2. *Let* h_i: $\mathbf{I} \to \mathbf{I}$, $i = 1, 2, \ldots,$ *be functions which assume only the values* 0 *and* 3^{-i}. *If* $h = \sum_{i=1}^{\infty} h_i$, *then* $v(h) \geqslant \frac{1}{6} \sum_{i=1}^{\infty} v(h_i)$. *In particular, if some* $h_i \notin \mathbf{BV}$, *then* $h \notin \mathbf{BV}$.

PROOF. Let p be a discontinuity of h_i which is not a discontinuity of h_j for $j < i$. Then p is a discontinuity of h, contributing a jump of at least $3^{-i} - \sum_{j>i} 3^{-j} = \frac{1}{2}3^{-i}$ in h, but contributing no more than $2 \sum_{j>i} 3^{-j} = 3^{-i+1}$ to $\sum v(h_i)$. If each $h_i \in \mathbf{BV}$, i.e. has only a finite number of discontinuities, the inequality follows. Otherwise some h_i has infinitely many discontinuities, and $v(h) = v(h_i) = \infty$.

THEOREM 3. *For* g: $\mathbf{I} \to \mathbf{I}$, *the composition* $f \circ g$ *belongs to* $\mathbf{BV}$ *for all* f *in* $\mathbf{BV}$ *iff* $g \in \mathbf{BV}(N)$ *for some* N.

PROOF. *Sufficiency*. Suppose f: $\mathbf{I} \to \mathbf{R}$ is increasing, and $g \in \mathbf{BV}(N)$. Then $f \circ g \in \mathbf{BV}'(N)$ because $f^{-1}([a, b]) = [f^{-1}(a), f^{-1}(b)]$, and hence $(f \circ g)^{-1}([a, b]) \in J_N$ for all a, b. Moreover, any $f \in \mathbf{BV}$ can be expressed as a difference of two increasing functions $f = f_1 - f_2$, and since $f_1 \circ g, f_2 \circ g \in \mathbf{BV}'(N)$, we have $f \circ g = f_1 \circ g - f_2 \circ g \in \mathbf{BV}$, using Lemma 1.

Necessity. Suppose for all N, $g \notin \mathbf{BV}(N)$. Find intervals $I_n \subseteq \mathbf{I}$, $n = 1, 2, \ldots,$ such that $g^{-1}(I_n) \notin J_{3^n}$, i.e. $g^{-1}(I_n)$ cannot be expressed as a union of 3^n intervals. Define $f_n = 3^{-n}\chi_{I_n}$, and $f = \sum_{n=1}^{\infty} f_n$. Since $v(f_n) \leqslant 2 \cdot 3^{-n}$, we have $v(f) \leqslant \sum v(f_n) \leqslant 2 \sum 3^{-n} < \infty$, so $f \in \mathbf{BV}$. On the other hand, $f_n \circ g = 3^{-n}\chi_{g^{-1}(I_n)}$, so $v(f_n \circ g) \geqslant 2 \cdot 3^{-n} \cdot 3^n = 2$. By Lemma 2, $v(f \circ g) = \infty$, i.e. $f \circ g \notin \mathbf{BV}$.

THEOREM 4. *For* g: $\mathbf{I} \to \mathbf{I}$, *the composition* $g \circ f$ *belongs to* $\mathbf{BV}$ *for all* f *in* $\mathbf{BV}$ *iff* g *satisfies a Lipschitz condition on* $\mathbf{I}$.

PROOF. *Sufficiency*. Suppose $|g(y) - g(z)| \leqslant M|y - z|$ for all $y, z \in \mathbf{I}$, and suppose $f \in \mathbf{BV}$ with $v(f) = L$. If $P = \{x_0, \ldots, x_n\}$ is a partition of $\mathbf{I}$,

$$\sum_P |\Delta(g \circ f)_i| \leqslant \sum_{i=1}^{n} M|f(x_i) - f(x_{i-1})| \leqslant LM,$$

so $g \circ f \in \mathbf{BV}$.

Necessity. If g does not satisfy a Lipschitz condition on $\mathbf{I}$, there exist $y_n'' < z_n''$ with $|g(y_n'') - g(z_n'')| > (n^2 + n)|y_n'' - z_n''|$ for any $n = 1, 2, \ldots$. Since $\mathbf{I}$ is compact, there is a convergent subsequence $\{y_n'\}$ of $\{y_n''\}$. Say $y_n' \to y$. Take a further subsequence $\{y_n\}$ with $|y - y_n| < (n + 1)^{-2}$. Take $\{z_n\}$ to be the corresponding subsequence of $\{z_n''\}$, and $\delta_n = |y_n - z_n|$. We have $|g(y_n) - g(z_n)| > (n^2 + n)\delta_n$, and hence $\delta_n < (n^2 + n)^{-1}$.

Define f by $f(0) = 0, f(1) = y_1$, while on $[(n + 1)^{-1}, n^{-1})$,

$$f(x) = \begin{cases} y_n & \text{if } x - (n + 1)^{-1} \text{ is a multiple of } \delta_n, \\ z_n & \text{otherwise.} \end{cases}$$

For a fixed n, let $m' = \delta_n^{-1}(n^2 + n)^{-1}$, and $m = \max\{k \in \mathbf{Z}: k < m'\}$. Since $m' > 1$, we have $m'/2 \leqslant m < m'$. Consider the partition P of $[(n+1)^{-1}, n^{-1}]$:

$$P = \left\{(n+1)^{-1} + \tfrac{1}{2}r\delta_n : r = 0, 1, \ldots, 2m\right\} \cup \{n^{-1}\}.$$

Then

$$\sum_P |\Delta(g \circ f)_i| \geqslant 2m|g(y_n) - g(z_n)| > m'(n^2 + n)\delta_n = 1,$$

so $v(g \circ f, (n+1)^{-1}, n^{-1}) > 1$: whence $g \circ f \notin \mathbf{BV}$.

On the other hand, since f is a step function on $[(n+1)^{-1}, n^{-1}]$,

$$\begin{aligned} v\big(f, (n+1)^{-1}, n^{-1}\big) &= 2m|y_n - z_n| + |y_{n-1} - t_n| \qquad \text{where } t_n = y_n \text{ or } z_n \\ &< 2m'\delta_n + |y_{n-1} - y| + |y - y_n| + |y_n - z_n| \\ &< 2(n^2 + n)^{-1} + n^{-2} + (n+1)^{-2} + n^{-2} < 5n^{-2}. \end{aligned}$$

Since $\sum n^{-2}$ converges, $f \in \mathbf{BV}$.

Chaika and Waterman [1] obtained a theorem analogous to our Theorem 4, but for certain classes larger than **BV**. Interestingly, they found the Lipschitz condition to be necessary and sufficient in those cases as well.

References

1. M. Chaika and D. Waterman, *On the invariance of certain classes of functions under composition*, Proc. Amer. Math. Soc. **43** (1974), 345–348. MR **48** #8704.

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