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THE SPACE OF RETRACTIONS OF A COMPACT Q-MANIFOLD IS AN l^2-MANIFOLD

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ABSTRACT. In this paper, we prove that the space of retractions of a compact Hilbert cube manifold is an l^2-manifold. This answers a question raised by T. A. Chapman.

Let M be a compact Q-manifold, and let $R(M)$ be the space of retractions of M, equipped with the sup-metric, i.e., $R(M) = \{e | e\colon M \to M \text{ is continuous}, e^2 = e\}$. T. A. Chapman [2] proved that $R(M)$ *is an* ANR, and he asked whether $R(M)$ is an l^2-manifold. The purpose of this paper is to answer this question affirmatively.

THEOREM. *$R(M)$ is an l^2-manifold.*

Recently, H. Toruńczyk gave the mapping characterization of l^2-manifolds [3, Corollary 3.3], which states that a separable complete-metrizable ANR X is an l^2-manifold if and only if the following two conditions are satisfied:

(*) For each $n \in \mathbf{N}$, any two continuous maps $f, g\colon I^n \to X$ can be arbitrarily closely approximated by continuous maps with disjoint images.

(**) For any sequence $\{P_n\}_{n\in\mathbf{N}}$ of compact polyhedra, any continuous map $f\colon \sum_{n\in\mathbf{N}} P_n \to X$ can be arbitrarily closely approximated by a continuous map $g\colon \sum_{n\in\mathbf{N}} P_n \to X$ such that $\{g(P_n)\}_{n\in\mathbf{N}}$ is locally finite in X.

Actually, the condition (*) is unnecessary,[1] that is, the condition (**) implies the condition (*) since, as noted in [4], if $g_i\colon I^n \to X$ ($i \in \mathbf{N}$) are approximations of a continuous map $g\colon I^n \to X$ such that $\{g_i(I^n)\}_{i\in\mathbf{N}}$ is locally finite in X, then for every compact subset K of X, $g_i(I^n)$ is disjoint from K for almost all $i \in \mathbf{N}$. Thus, it suffices to show that $R(M)$ satisfies the condition (**).

Since M is homeomorphic to $M \times Q$ [1], we may show that *$R(M \times Q)$ satisfies the condition* (**). Points of $M \times Q$ will be denoted by $y = (y_0, y_1, y_2, \ldots)$, where $y_0 \in M$ and $y_i \in I_i = [-1, 1]$ $(i = 1, 2, \ldots)$. We use the metric on $M \times Q$ defined by

$$d(y, y') = d_M(y_0, y_0') + \sum_{i=1}^{\infty} 2^{-i}|y_i - y_i'|,$$

where d_M is a metric on M. $R(M \times Q)$ is equipped with the sup-metric $d(e, e') = \sup\{d(e(y), e'(y)) | y \in M \times Q\}$.

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[1]D. W. Curtis suggested to me that J. van Mill noted this.

An ambient invertible isotopy h_t $(t \in [1, \infty))$. We will define an ambient invertible isotopy $h_t\colon (M \times Q) \times Q \to M \times Q$ $(t \in [1, \infty))$. First we define homeomorphisms h_i, $h_{i+j/i}\colon (M \times Q) \times Q \to M \times Q$ $(i = 1, 2, \ldots;\ j = 1, \ldots, i-1)$ as follows.

$$
\begin{aligned}
h_1(y, z) &= (y_0, y_1, z_1, y_2, z_2, y_3, z_3, y_4, z_4, \ldots),\\
h_2(y, z) &= (y_0, y_1, y_2, -z_1, z_2, y_3, z_3, y_4, z_4, \ldots),\\
h_{2+(1/2)}(y, z) &= (y_0, y_1, y_2, -z_1, y_3, -z_2, z_3, y_4, z_4, \ldots),\\
h_3(y, z) &= (y_0, y_1, y_2, y_3, z_1, -z_2, z_3, y_4, z_4, \ldots),\\
h_{3+(1/3)}(y, z) &= (y_0, y_1, y_2, y_3, z_1, -z_2, y_4, -z_3, z_4, \ldots),\\
h_{3+(2/3)}(y, z) &= (y_0, y_1, y_2, y_3, z_1, y_4, z_2, -z_3, z_4, \ldots),\\
h_4(y, z) &= (y_0, y_1, y_2, y_3, y_4, -z_1, z_2, -z_3, z_4, \ldots),\\
&\ \vdots\\
h_i(y, z) &= \big(y_0, \ldots, y_i, (-1)^{i-1} z_1, (-1)^{i-2} z_2, \ldots, (-1) z_{i-1}, z_i,\\
&\qquad y_{i+1}, z_{i+1}, y_{i+2}, z_{i+2}, \ldots\big),\\
&\ \vdots\\
h_{i+(j/i)}(y, z) &= \big(y_0, \ldots, y_i, (-1)^{i-1} z_1, \ldots, (-1)^j z_{i-j}, y_{i+1},\\
&\qquad (-1)^j z_{i-j+1}, \ldots, (-1) z_i, z_{i+1}, y_{i+2}, z_{i+2}, \ldots\big),\\
&\ \vdots\\
h_{i+1}(y, z) &= \big(y_0, \ldots, y_{i+1}, (-1)^{(i+1)-1} z_1, \ldots, (-1) z_i, z_{i+1},\\
&\qquad y_{i+2}, z_{i+2}, y_{i+3}, z_{i+3}, \ldots\big)\\
&\ \vdots
\end{aligned}
$$

Let $\theta_t\colon [-1, 1]^2 \to [-1, 1]^2$ $(t \in I)$, be an ambient invertible isotopy such that $\theta_0 = \mathrm{id}$, $\theta_1(s_1, s_2) = (s_2, -s_1)$ for each $(s_1, s_2) \in [-1, 1]^2$. For each

$$t \in [i + (j-1)/i, i + j/i] \quad (i = 1, 2, 3, \ldots; j = 1, \ldots, i),$$

we define

$$h_t = \theta^{i,j}_{i(t-i-(j-1)/i)} \circ h_{i+(j-1)/i}\colon (M \times Q) \times Q \to M \times Q,$$

where $\theta_t^{i,j}\colon M \times Q \to M \times Q$ $(t \in I)$ is an ambient invertible isotopy defined by

$$\theta_t^{i,j}(y) = (y_0, \ldots, y_{2i-j}, \theta_t(y_{2i-j+1}, y_{2i-j+2}), y_{2i-j+3}, \ldots).$$

Note that our ambient invertible isotopy h_t $(t \in [1, \infty))$ has the following properties:

(1) If $t < i$, then $p_{2i} h_t(y, z) = z_i$ for all $(y, z) \in (M \times Q) \times Q$,

(2) If $t > i$, then $p_i h_t(y, z) = y_i$ for all $(y, z) \in (M \times Q) \times Q$.

PROOF OF THE CONDITION (**). Let $\{P_n\}_{n\in\mathbf{N}}$ be a sequence of compact polyhedra and f: $\Sigma_{n\in\mathbf{N}} P_n \to R(M \times Q)$ a continuous map. For any continuous function ε: $R(M \times Q) \to (0, \infty)$, there exists a continuous function δ: $r(M \times Q) \to (0, 1]$ such that $\delta(e) \leqslant \varepsilon(e)/2$ and $d(y, y') < \delta(e)$ $(y, y' \in M \times Q)$ implies $d(e(y), e(y'))$ $< \varepsilon(e)/2$ for each $e \in R(M \times Q)$. (This is because the function $\bar{\delta}$: $R(M \times Q) \to$ $(0, \infty)$ defined by $\bar{\delta}(e) = \sup\{\delta > 0 | d(y, y') < \delta \Rightarrow d(e(y), e(y')) < \varepsilon(e)/2\}$ is lower semicontinuous.)

Each P_n admits a triangulation K_n such that

$$\sup\{\delta f(x) | x \in \sigma\} - \inf\{\delta f(x) | x \in \sigma\} < 2^{-n} \tag{3}$$

and

$$\sup\{\delta f(x) | x \in \sigma\} < 2 \inf\{\delta f(x) | x \in \sigma\} \tag{4}$$

for each simplex σ of K_n. For each $x \in P_n$, let $(x(v))_{v\in K_n^0}$ be the barycentric coordinates of x with respect to the triangulation K_n. For each vertex v of K_n, choose a positive integer $i(v)$ so that $2^{-i(v)+2} \leqslant \delta f(v) < 2^{-i(v)+3}$ and define a continuous function t: $\Sigma_{n\in\mathbf{N}} P_n \to [1, \infty)$ by

$$t(x) = \sum_{v\in K_n^0} x(v) i(v) \quad \text{for each } x \in P_n.$$

For each $n \in \mathbf{N}$, let r_n: $[-1, 1] \to [-1, 1]$ be a piecewise-linear map such that $r_n(-1) = r_n(0) = r_n(1) = 1$ and $r_n(1/n) = 1/n$. Then define r_n^*: $Q \to Q$ by

$$r_n^*(z_1, z_2, \dots) = (r_n(z_1), r_n(z_2), \dots).$$

Now, we define a map g: $\Sigma_{n\in\mathbf{N}} P_n \to R(M \times Q)$ by

$$g(x) = h_{t(x)} \circ (f(x) \times r_n^*) \circ h_{t(x)}^{-1} \quad \text{for each } x \in P_n.$$

Since h_t $(t \in [1, \infty))$ is an ambient invertible isotopy, g is continuous. We assert that g is a desired approximation of f.

First, we see that $d(f(x), g(x)) < \varepsilon f(x)$ for each $x \in \Sigma_{n\in\mathbf{N}} P_n$. Let $x \in \sigma \in K_n$. There exists a vertex v of σ such that $i(v) \leqslant t(x)$. From (2) and (4), $d(h_{t(x)}, p) \leqslant$ $2^{-i(v)+1} \leqslant \delta f(v)/2 < \delta f(x)$ where p: $(M \times Q) \times Q \to M \times Q$ is the projection. Hence

$$d(f(x)p, f(x)h_{t(x)}) < \varepsilon f(x)/2.$$

Thus

$$\begin{aligned} d(f(x), g(x)) &\leqslant d\big(f(x), p(f(x) \times r_n^*)h_{t(x)}^{-1}\big) + d\big(p(f(x) \times r_n^*)h_{t(x)}^{-1}, g(x)\big) \\ &= d\big(f(x)h_{t(x)}^{-1}, p(f(x) \times r_n^*)\big) + d(p, h_{t(x)}) \\ &< d\big(f(x)h_{t(x)}^{-1}, f(x)p\big) + \delta f(x) < \varepsilon f(x). \end{aligned}$$

Next, we claim that $\{g(P_n)\}_{n\in\mathbf{N}}$ is locally finite in $R(M \times Q)$. Suppose not. Then there exists a convergent sequence $g(x_{n_i}) \to e \in R(M \times Q)$, where $x_{n_i} \in P_{n_i}$ for each $i \in \mathbf{N}$. For convenience, assume $n_i = n$; thus $g(x_n) \to e$ $(n \to \infty)$, $x_n \in P_n$. If there exists a positive integer i_0 such that $t(x_n) \leqslant i_0$ for each $n \in \mathbf{N}$, then $p_{2i_0} g(x_n) = r_n p_{2i_0}$ from (1), where p_i: $M \times Q \to I_i$ is the projection. This is a contradiction, because $p_{2i_0}g(x_n) \to p_{2i_0}e$ but $r_n p_{2i_0}$ cannot converge to any continuous function. Thus $\{t(x_n)\}_{n\in \mathbf{N}}$ is unbounded. Hence we may assume that $t(x_n) \to \infty$ $(n \to \infty)$. Then $h_{t(x_n)} \to p$, so $d(g(x_n), p(f(x_n) \times r_n^*)h_{t(x_n)}^{-1}) \to 0$. Hence $p(f(x_n) \times r_n^*)h_{t(x_n)}^{-1} \to e$, so $d(p(f(x_n) \times r_n^*), eh_{t(x_n)}^{-1}) \to 0$. Since $eh_{t(x_n)} \to ep$, $f(x_n)p = p(f(x_n) \times r_n^*) \to ep$. Therefore $f(x_n) \to e$ because p is onto. On the other hand, there are vertices v_n of the carriers of x_n such that $t(x_n) \leqslant i(v_n)$. Since $\delta f(v_n) < 2^{-i(v_n)+3}$ and $t(x_n) \to \infty$, $\delta f(v_n) \to 0$. From (3), $|\delta f(v_n) - \delta f(x_n)| < 2^{-n}$, then $\delta f(x_n) \to 0$. Hence $\delta(e) = 0$. This is a contradiction. $\square$

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