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ON THE FIELD OF A 2-BLOCK

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ABSTRACT. For a p-block B satisfying some conditions, a field $Q(B)$ is defined. It is proved that for a 2-block B of a finite metabelian group G, $Q(B) = Q(\theta)$ for some irreducible character θ in B if the 2-Sylow subgroup P of the commutator group G' is cyclic. This is shown to be false in general.

1. Introduction. For a prime p a p-block B of a finite group G is defined to be real if it contains the complex conjugate of one (and hence every one) of its irreducible ordinary characters (over the complex field $\mathbf{C}$). In §2 of this paper we define the field $Q_0(B)$ when some conditions on B are satisfied, where Q_0 is an extension of the rational field Q. If $Q_0 = \mathbf{R}$, the real field, then $\mathbf{R}(B)$ is defined for all p-blocks B and $\mathbf{R}(B) = \mathbf{R}$ if B is real and $\mathbf{R}(B) = \mathbf{C}$ otherwise. These notations and a result of Gow [5] imply that every 2-block B of a finite group G contains an irreducible (ordinary) character θ such that $\mathbf{R}(B) = \mathbf{R}(\theta)$. When $Q_0 = Q$ we show that $Q(B)$ is defined when B is a p-block of a finite metabelian group. We investigate the equality $Q(B) = Q(\theta)$ for some irreducible character θ of the 2-block B. In §3 we show that this equality holds (for some $\theta \in B$) if the commutator group G' is abelian and the 2-Sylow subgroup of G' is cyclic. In §4 we give an example in which this equality does not hold (for any $\theta \in B$).

2. The definition of $Q_0(B)$. Let G be a finite group and let B be a p-block of G. For any ordinary character χ of G let $Q_0(\chi) = Q_0(\chi(g) | g \in G)$, Q_0 an extension of the rational field Q, and let $F = Q_0(\chi | \chi$ an irreducible ordinary character in $B)$. Since $F \subseteq Q_0(\zeta)$ for some root of unity ζ, it follows that F and $Q_0(\chi)$ are normal extensions of Q_0. Let $\mathcal{G} = \mathcal{G}(F/Q_0)$ be the Galois group of F over Q_0. For each $\tau \in \mathcal{G}$ and ordinary character χ of G, define χ^τ by $\chi^\tau(g) = \chi(g)^\tau$ for all $g \in G$. Let $S(\tau) = \{\chi^\tau | \chi$ an irreducible ordinary character of $B\}$. In order to be able to define the field $Q_0(B)$, we shall assume that for each $\tau \in \mathcal{G}$, $S(\tau)$ is the set of all irreducible ordinary characters of some p-block of G, and we appropriately denote this p-block by B^τ. Let χ be an irreducible ordinary character of B, $\mathcal{H}(\chi) = \{\tau \in \mathcal{G} | \chi^\tau = \chi\}$ and $\mathcal{H}(B) = \{\tau \in \mathcal{G} | B^\tau = B\}$. It is easy to see that $Q_0(\chi)$ is the fixed field of $\mathcal{H}(\chi)$ and that $\mathcal{H}(\chi) \subseteq \mathcal{H}(B)$ for all irreducible ordinary characters χ of B. Define the field $Q_0(B)$ of the p-block B as the subfield of F fixed by $\mathcal{H}(B)$.

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When $Q_0(B)$ is defined it follows that $Q_0(B) \subseteq \bigcap Q_0(\chi)$ where the intersection is over all irreducible ordinary characters of B. Also note that we would have gotten the same field $Q_0(B)$ if we had taken $Q_0(\zeta)$, as we shall often do, instead of F.

Let $Q_0 = \mathbf{R}$, the real field, and let χ^c be the complex conjugate of the character χ. Since the complex conjugate of a Brauer character is also a Brauer character, the set $\{\chi^c | \chi$ an irreducible ordinary character of $B\}$, B some p-block of G, forms the irreducible ordinary characters of a p-block B^c which is known as the complex conjugate p-block of B. Thus $\mathbf{R}(B)$ is always defined. Brauer [**3**, VIII] defines B to be real if $B^c = B$. This means that $\mathbf{R}(B) = \mathbf{R}$ or $\mathcal{K}(B) = \mathcal{G}$. [Note that $\mathcal{G}$, in this case, is of order 1 or 2.] Gow [**5**, (5.1)] showed that every real 2-block contains a real irreducible ordinary character. From this and the above notations it follows that if B is a 2-block of a finite group then there exists an irreducible ordinary character θ in B such that $\mathbf{R}(B) = \mathbf{R}(\theta)$. This equality need not be true for p-blocks where p is odd. To study $Q(B)$, when this field is defined, we need

LEMMA. *Let G be a finite metabelian group and let B be a p-block of G. Then $Q(B)$ is defined.*

PROOF. Let L be a subgroup of G', $K(L)$ a subgroup of $N(L)$, the normalizer of L in G, $K(L) \supseteq G'$, and $K(L)/L$ a maximal abelian subgroup of $N(L)/L$. If G'/L is cyclic, and T' is a linear ordinary representation of $K(L)$ with $\ker T' \cap G' = L$, then T'^G is irreducible and every irreducible ordinary representation of G is given in this form. See [**1**, §3]. Let p be a prime, H a subgroup of G', G'/H cyclic, and $p \nmid |G'/H|$. If T' is a linear representation of $K(H)$, $\ker T' \cap G' = H$, then, from [**2**, §3], the p-modular representation $\overline{T}'^G$ is irreducible and every irreducible p-modular representation of G is given in this form. (The notations are the same as in [**2**].) The Brauer character χ_B of $\overline{T}'^G$ is the restriction of the character χ of T'^G to the p-regular elements of G. Let $\tau \in \mathcal{G} = \mathcal{G}(F/Q)$; then χ^τ is the character of $(T'^\tau)^G$, or $(\chi_B)^\tau$ is also a Brauer character. Now let T' be a linear representation of $K(L)$, $\ker T' \cap G' = L$ (p may divide $|G'/L|$), and Z a linear representation of $K(L)$ such that $Z(k) = T'(k)$ if k is p-regular and $Z(k) = 1$ if k is p-singular. Then $\ker Z \cap G' = H \supseteq L$ and $p \nmid |G'/H|$. Pick $K(H) \supseteq K(L)$. (Note that Z^G need not be irreducible.) Thus $\overline{Z}^G$ and $\overline{T}'^G$ have the same Brauer characters and the irreducible composition factors of $\overline{Z}^G$ and $\overline{T}'^G$ are precisely the $|K(H)/K(L)|$ representations $\overline{Z}_i^G$, where the Z_i's are the extensions of Z to $K(H)$. Now let B be a p-block of G, $\tau \in \mathcal{G} = \mathcal{G}(F/Q)$, and $S(\tau)$ be defined as above. Let χ and θ be irreducible ordinary characters in B and assume the Brauer characters χ_B and θ_B (which are the restrictions of χ and θ, respectively, to the p-regular elements of G) have in common (when written as a summation of irreducible Brauer characters) the irreducible Brauer characters $\mu_1, \ldots, \mu_n$ (these need not necessarily be distinct). Then χ_B^τ and θ_B^τ will have in common the Brauer characters $\mu_1^\tau, \ldots, \mu_n^\tau$. Thus $S(\tau)$ is the set of all irreducible ordinary characters of some p-block B^τ. Thus the field $Q(B)$ is defined and the proof is complete.

3. The main result. We prove

THEOREM. *Let B be a 2-block of a finite metabelian group G and let P be the 2-Sylow subgroup of G'. Assume P is cyclic. Then there exists an irreducible 2-rational character θ in B of height* 0 *such that $Q(B) = Q(\theta)$.*

PROOF. From §2, $Q(B)$ is defined. We use the results in **[1]** and **[2]**. Let $G' = P \times G_1$ and $H = P \times \Lambda$, $\Lambda \subseteq G_1$, such that G'/H is cyclic. Let $G_1/\Lambda = \langle b\Lambda \rangle$, $P = \langle a \rangle$; then $G'/H = \langle bH \rangle$. Let $\Lambda \subseteq L \subseteq H$, L a subgroup of G'. Then G'/L is cyclic and $L = \langle \Lambda, a^{2^r} \rangle$. Let $x \in N(H)$, the normalizer of H in G. Then $x^{-1}Px = P$ and $x^{-1}\Lambda x = \Lambda$, and since P is cyclic we have $x^{-1}Lx = L$, or $N(H) \subseteq N(L)$. Conversely if $x \in N(L)$, then $x^{-1}\Lambda x = \Lambda$ or $x^{-1}Hx = H$. Thus we have $N(H) = N(L) = N(\Lambda) = N$, for all L, $\Lambda \subseteq L \subseteq H$. For any subgroup L of G' define $K(L)$ as done above and if $\Lambda \subseteq L \subseteq H$, then choose $K(\Lambda) \subseteq K(L) \subseteq K(H)$. Let $x \in K(H) - K(\Lambda)$ and assume $xK(\Lambda)$ is of odd order m in $K(H)/K(\Lambda)$. From the definitions of $K(H)$ and $K(\Lambda)$, $x^{-1}yx \equiv y \pmod{\Lambda}$ for all $y \in G_1$. Assume $x^{-1}ax \equiv a^r \pmod{\Lambda}$, $r \not\equiv 1 \pmod{|P|}$. Then x must be of even order. Thus $x^{-1}ax \equiv a \pmod{\Lambda}$, or $x^{-1}yx \equiv y \pmod{\Lambda}$ for all $y \in G'$. Let $y \in K(\Lambda) - G'$ and assume $x^{-1}yx \equiv yc \pmod{\Lambda}$, $c \in G'$. Then $c = a^{\lambda}c_1$, $c_1 \in G_1$. Assume $c_1 \notin \Lambda$; then $c_1 \notin H$ and hence $x^{-1}yx \not\equiv y \pmod{H}$, a contradiction since $K(H)/H$ is abelian. Thus $c_1 \in \Lambda$, and $x^{-1}yx \equiv ya^{\lambda} \pmod{\Lambda}$. Since $x^m \in K(\Lambda)$, and x commutes with a, we have $m\lambda \equiv 0 \pmod{|P|}$ or $x^{-1}yx \equiv y \pmod{\Lambda}$ for all $y \in K(\Lambda)$. But $K(\Lambda)/\Lambda$ is a maximal abelian subgroup of N/Λ, and hence $x \in K(\Lambda)$. Thus $K(H)/K(\Lambda)$ is an (abelian) 2-group.

Let σ be a linear 2-modular representation of $K(\Lambda)$ such that $\ker \sigma \cap G' = H$ and let $S = \ker \sigma$. Let $B(\sigma, H)$ be the collection of all representations T'^G where T' is a linear representation of $K(L)$, $\ker T' \cap G' = L$, with the modular representation $\overline{T}'_{K(\Lambda)}$ being G-conjugate to σ, and L runs over all subgroups of G', $\Lambda \subseteq L \subseteq H$. Include in $B(\sigma, H)$ the composition factors (and their Brauer characters) of the modular representations $\overline{T'^G}$, and the characters of T'^G. From **[2, §4]** $B(\sigma, H)$ is a 2-block and every 2-block of G is given in this form. Let V be a linear ordinary representation of $K(H)$ such that $V(g) = 1$ if g is 2-singular and $\overline{V}_{K(\Lambda)} = \sigma$. Then V^G is irreducible and $V^G \in B(\sigma, H)$. Let θ be the character of V^G; then θ is 2-rational and is of height 0 in $B(\sigma, H)$. We shall prove that $Q(B(\sigma, H)) = Q(\theta)$.

We claim that $I(\sigma) = I(V) = K(H)$, where $I(\sigma)$ and $I(V)$ are the inertia groups of σ and V, respectively. Since $K(\Lambda)/S$ is of odd order, $K(H)/S = K(\Lambda)/S \times M/S$, where M/S is the 2-Sylow subgroup of $K(H)/S$. Let $S_1 = \ker V$; then $S_1 = \langle S, M \rangle$, and it can easily be shown that $N(S) = N(S_1) \subseteq N(H)$. Let $x \in I(V)$; then $x \in N(S_1)$ and $x^{-1}kx \equiv k \pmod{S_1}$ for all $k \in K(H)$. Since $x^{-1}kx \equiv kc \pmod{H}$ for some $c \in G'$, it follows that $c \in S_1$. Hence, $c \in S_1 \cap G' = H$ or $x^{-1}kx \equiv k \pmod{H}$ for all $k \in K(H)$, i.e., $x \in K(H)$. Therefore, $I(V) = K(H)$. Assume $x \in I(\sigma)$; then $x \in N(S)$ and $x^{-1}kx \equiv k \pmod{S}$ for all $k \in K(\Lambda)$. As above, this means that $x^{-1}kx \equiv k \pmod{H}$ for all $k \in K(\Lambda)$. Let M_1/H be the

2-Sylow subgroup of $K(H)/H$; then $K(H)/H = \langle K(\Lambda)/H, M_1/H\rangle$. Since $N(S) \subseteq N(H)$, x normalizes M_1. Let $k \in M_1$; then $x^{-1}kx \equiv kc \pmod{H}$ where $c \in G'$. Assume $c \notin H$. Then since $|G'/H|$ is odd, we have c of odd order modulo H. But k is of order a power of 2 modulo H, and thus kc is not of order a power of 2 modulo H. This contradiction implies that $c \in H$. Therefore $I(\sigma) = K(H)$, proving our claim.

Thus the number of G-conjugates of σ having kernel S is equal to the number of G-conjugates of V having kernel S_1 and this number equals $|N(S)/K(H)|$. Thus if σ_1 is G-conjugate to σ, $\ker \sigma_1 = \ker \sigma = S$, and $(\overline{V}_1)_{K(\Lambda)} = \sigma_1$, $\ker V_1 = \ker V = S_1$; then V_1 is G-conjugate to V.

Now let ζ be a primitive $|G|$th root of unity; then $Q(\chi) \subseteq Q(\zeta)$ for all $\chi \in B(\sigma, H)$. Let $\tau \in \mathcal{G} = \mathcal{G}(Q(\zeta)/Q)$ and assume $B(\sigma, H)^\tau = B(\sigma, H)$. This means that $(V^\tau)^G \in B(\sigma, H)$ and, hence, from [2, §4], $(\overline{V^\tau})_{K(\Lambda)}$ is G-conjugate to σ. But $\ker V^\tau = \ker V = S_1$, and thus V^τ is G-conjugate to V. From [4, (45.6)], it follows that V^G and $(V^\tau)^G$ are equivalent or $\theta^\tau = \theta$. Thus $\mathcal{H}(B(\sigma, H)) = \mathcal{H}(\theta)$ and, hence, $Q(B(\sigma, H)) = Q(\theta)$, completing the proof.

In the above result we used only the fact that 2 is the smallest prime dividing the order $|G|$ of G.

COROLLARY. *Let p be the smallest prime dividing the order $|G|$ of the metabelian group G and let B be a p-block of G. Assume the p-Sylow subgroup P of G' is cyclic. Then there exists an irreducible p-rational character θ in B of height* 0 *such that $Q(B) = Q(\theta)$.*

COROLLARY. *Every* 2*-block B of a finite metacyclic group contains an irreducible* 2*-rational character θ of height* 0 *such that $Q(B) = Q(\theta)$.*

4. An example. We give a 2-block B of a group G with the property that $Q(B) \subset Q(\chi)$ for all irreducible ordinary characters χ of B. Again we make use of the results in [1] and [2].

Let p and q be odd primes, $p|q-1$, $n = p^2$ and $\lambda > 3$. Consider the group G generated by $a_1, a_2, \ldots, a_n, b, c$ with the defining relations $a_i^2 = b^q = c^{p^\lambda} = 1$, $a_ia_j = a_ja_i$, $a_ib = ba_i$, $i, j = 1, 2, \ldots, n$, $c^{-1}a_ic = a_{i+1}$, $i = 1, 2, \ldots, n-1$, $c^{-1}a_nc = a_1$, and $c^{-1}bc = b^r$ with $r \not\equiv 1 \pmod{q}$ and $r^p \equiv 1 \pmod{q}$. Then $G' = \langle a_1a_2, a_2a_3, \ldots, a_{n-1}a_n, b\rangle$ is of order $2^{n-1}q$. Let $P = \langle a_1, \ldots, a_n\rangle$, $G_1 = \langle P, b\rangle$, and for convenience take H a subgroup of G_1. Let $H = P$; then $K(P) = \langle P, b, c^p\rangle$, $G_1/P = \langle bP\rangle$. Let $\Lambda = 1$; then $K(\Lambda) = \langle P, b, c^{p^2}\rangle$. Define the modular linear representation σ of $K(\Lambda)$ by $\sigma(b)$ a primitive qth root of unity and $\sigma(c^{p^2})$ a primitive $p^{\lambda-2}$th root of unity. If L is a subgroup of G_1 such that G_1/L is cyclic and $\Lambda \subseteq L \subseteq P$, then $K(L) = K(\Lambda)$ or $K(L) = K(H)$. Both of these cases occur, the first when $L = \langle a_2, \ldots, a_n\rangle$ and the latter when $L = P$. The ordinary irreducible representations in the 2-block $B(\sigma, P)$ consist of all T'^G, where T' is a linear representation of $K(L)$, $\ker T' \cap G_1 = L$, for all L such that $\Lambda \subseteq L \subseteq P$, G_1/L cyclic and $\overline{T}'_{K(\Lambda)}$ being G-conjugate to σ. Thus if ζ is a primitive $p^{\lambda-1}$ qth root of unity (over Q), then $Q(\chi) \subseteq Q(\zeta)$ for all ordinary irreducible characters $\chi \in B(\sigma, P)$. We shall study $Q(B(\sigma, P))$ and $Q(\chi)$ for all ordinary irreducible characters χ in $B(\sigma, P)$.

First assume $c^{-1}Lc = L$; then $K(L) = K(P)$ and $T'(c^p)$ is a $p^{\lambda-1}$th root of unity. Now $\langle K(P)/L \rangle$ is of order $p^{\lambda-1}q$ or $2p^{\lambda-1}q$ and each T' has exactly p G-conjugates all having the same kernel and exactly $\varphi(p^{\lambda-1}q)$ Q-conjugates, i.e. conjugates over Q, not all in $B(\sigma, P)$, having the same kernel. Thus if χ is the character of T'^G, then χ has exactly $\varphi(p^{\lambda-1}q)/p = \varphi(p^{\lambda-2}q)$ distinct Q-conjugates, and thus $[Q(\chi): Q] = \varphi(p^{\lambda-2}q)$. Now assume $c^{-1}Lc \neq L$ but $c^{-p}Lc^p = L$. Then $K(L) = K(P)$ and $|K(P)/L| = 2p^{\lambda-1}q$. Defining T' as above, it follows that all the p G-conjugates of T' have different kernels. Thus the character χ of T'^G has $\varphi(p^{\lambda-1}q)$ distinct Q-conjugates or $[Q(\chi): Q] = \varphi(p^{\lambda-1}q)$. Finally assume $c^{-p}Lc^p \neq L$. Then $K(L) = K(\Lambda)$ and $|K(L)/L| = 2p^{\lambda-2}q$. Define T' of $K(L)$ such that $T'(c^{p^2})$ is a $p^{\lambda-2}$th root of unity with $\overline{T}'$ being G-conjugate to σ. Then T' has p^2 G-conjugates, all of which have different kernels, and it has $\varphi(p^{\lambda-2}q)$ Q-conjugates, not all in $B(\sigma, H)$, having the same kernel. Thus the character χ of T'^G has $\varphi(p^{\lambda-2}q)$ distinct Q-conjugates or $[Q(\chi): Q] = \varphi(p^{\lambda-2}q)$ and $Q(\chi) = Q(T'(d))$ where $K(L)/L = \langle dL \rangle$. We shall show that $Q(B(\sigma, P))$ is a proper subfield of $Q(T'(d))$.

Let $\mathcal{G} = \mathcal{G}(Q(T'(d))/Q)$ and $\tau \in \mathcal{G}$, $\tau \neq 1$. Then T'^G and $(T'^\tau)^G$ are inequivalent and they belong to the same 2-block $B(\sigma, P)$ if and only if $\overline{T}'$ and $\overline{T'^\tau}$ are both G-conjugate to σ. Assuming $\overline{T}' = \sigma$, and since the inertia group of σ is $K(H)$, p of the representations $\overline{T'^\tau}$ are G-conjugate to σ. Thus $(T'^\tau)^G \in B(\sigma, P)$, i.e., $B(\sigma, P)^\tau = B(\sigma, P)$ for some $\tau \neq 1$. Since $\chi \neq \chi^\tau$, χ^τ the character of $(T'^\tau)^G$, we have $Q(B(\sigma, P))$ a proper subfield of $Q(T'(d)) = Q(\chi) = Q(\chi^\tau)$. In fact $[Q(B(\sigma, P)): Q] = \varphi(p^{\lambda-2}q)/p$. Since $[Q(\chi): Q] \geqslant \varphi(p^{\lambda-2}q)$ for all irreducible ordinary characters χ of $B(\sigma, P)$, our claim is proved.

We conclude with some remarks concerning the 2-block $B(\sigma, P)$ described in the above example. $B(\sigma, P)$ has p irreducible modular representations given by σ_i^G, $i = 1, \ldots, p$, where σ_i's are the extensions of σ to $K(P)$. If $c^{-p}Lc^p = L$, then $\overline{T}'^G$ is equivalent to some σ_i^G and for each i there is at least one T'^G such that $\overline{T}'^G$ is equivalent to σ_i^G. If $c^{-p}Lc^p \neq L$ then each σ_i^G, $i = 1, 2, \ldots, p$, appears once in the composition factors of each $\overline{T}'^G$.

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