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ZERO-FREE REGIONS FOR EXPONENTIAL SUMS

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ABSTRACT. If the sum of the exponentials of the components of a complex n-vector $P = (z_1, \ldots, z_n)$ vanishes, then P is at least $[1 + o(1)]\ln n$ from the diagonal of complex n-space, and this is essentially best possible.

We shall establish an analogue in several complex variables of the fundamental fact that

$$\exp z \neq 0 \tag{1}$$

for all complex z. Our contribution is one of formulation; the proof is quite simple. Define the distance $d(A, P)$ between any points

$$A = (a_1, \ldots, a_n), \qquad P = (z_1, \ldots, z_n), \tag{2}$$

in complex n-space by

$$d^2(A, P) = \sum |a_j - z_j|^2. \tag{3}$$

The *diagonal* of complex n-space is the set of complex n-tuples having all components identical.

THEOREM. *Let $n \geq 2$. If*

$$\sum_{j=1}^{n} \exp z_j = 0 \tag{4}$$

then $P = (z_1, \ldots, z_n)$ has distance at least

$$d_n = \left(1 + \tfrac{1}{n}\right)\ln n \tag{5}$$

from the diagonal of complex n-space. On the other hand, the sum of (4) *vanishes at a point whose distance from the diagonal is at most*

$$\left[1 + O(\ln n)^{-2}\right]\ln n. \tag{6}$$

REMARKS. The nonvanishing of the sum of (4) on the diagonal is simply (1). Since the point on the diagonal closest to $P = (z_1, \ldots, z_n)$ is easily shown to be $Q = (q, q, \ldots, q)$ where

$$q = \left(\sum z_j\right)/n, \tag{7}$$

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our result says that the exponential sum is nonzero provided the z_j exhibit only "small deviations from the mean". The power sums occurring in the present proof (and the P_1 of (13) used to show that the result is "essentially best possible") remind one of the power sums (and the point P_1 at which they are maximal) studied in Lakshmanamurti **[2]**. Lakshmanamurti extends previous work of K. Pearson and others on estimates of sums related to skewness, kurtosis, and like quantities. Some small refinement of our distance estimates might conceivably follow from a complex analogue of **[2]**.

PROOF. As above, let Q be the diagonal point closest to P, and set

$$W = P - Q = (w_1, \ldots, w_n). \tag{8}$$

The sum of the w_j vanishes by (7), so

$$0 = \left| \sum_j \sum_{m=0}^{\infty} \frac{w_j^m}{m!} \right| \geqslant n - \sum_{m=2}^{\infty} \left| \sum_j w_j^m \right| / m!. \tag{9}$$

Jensen's theorem **[1**, p. 28] asserts that for $p > 0$ any sum

$$\left(\sum |a_j|^p \right)^{1/p} \tag{10}$$

is decreasing in p. If (5) is false, then for $m \geqslant 2$ we have

$$\left| \sum w_j^m \right|^{2/m} \leqslant \sum |w_j|^2 = |P - Q|^2 < d_n^2. \tag{11}$$

Hence

$$\begin{aligned} 0 &> n - [-1 - d_n + \exp d_n] \\ &= n + 1 + (1 + n^{-1})\ln n - n^{1+(1/n)} = 1 + O(n^{-1} \ln^2 n), \end{aligned} \tag{12}$$

a contradiction for n large. We suppress the routine estimates that show (12) is false for $n \geqslant 2$.

Finally, to show that the zero-free "tube" about the diagonal has radius essentially no larger than $\ln n$, define $P_1 = (z_1, \ldots, z_n)$ by

$$\begin{aligned} z_1 &= (1 - n^{-1})[i\pi + \ln(n-1)], \\ z_j &= -n^{-1}[i\pi + \ln(n-1)], \qquad 2 \leqslant j \leqslant n. \end{aligned} \tag{13}$$

Equation (4) is then evident. Here $Q = 0$, so

$$|P - Q|^2 = |P|^2 = \pi^2 + \ln^2(n-1) + O(n^{-1} \ln^2 n) \tag{14}$$

and

$$|P - Q| = (1 + .5\pi^2 \ln^{-2} n)\ln n + O(\ln^{-2} n). \tag{15}$$

This proves the theorem.

For $n = 2$ a zero point nearest to the diagonal is $(-i\pi/2, i\pi/2)$ with distance $\pi/\sqrt{2} = 2.2214 \ldots$. For $n = 3$ a possibly nearest zero point is

$$P = (\alpha, \bar{\alpha}, 0) \tag{16}$$

where $\alpha = .9933 + 1.7570i$, approximately; its distance to the diagonal is $2.6138 \ldots$. As in **[2]** an optimal (here nearest) point can be located with arbitrary precision by the method of Lagrange multipliers, but unlike [2] the resulting equations here are transcendental and rather unwieldy.

References

1. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1964.
2. M. Lakshmanamurti, *On the upper bound of* $\sum_{i=1}^{n} x_i^m$ *subject to the conditions* $\sum x_i = 0$ *and* $\sum x_i^2 = n$, Math. Student **18** (1950), 111–116.

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