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BANACH SPACES WHICH ALWAYS CONTAIN SUPREMUM-ATTAINING ELEMENTS

PETER D. MORRIS

ABSTRACT. It is proved that if a weakly compactly generated Banach space X has the property that, for every closed, bounded convex subset K of X^*, there exists a nonzero element of X which attains its supremum on K, then X contains no copy of l^1.

Let X be a Banach space and let A be a bounded subset of its dual. Define

$$SA(A) = \{x \in X \setminus \{0\}: \hat{x} \text{ attains its supremum on } A\}.$$

Here $\hat{x}$ denotes the image of x under the natural embedding of X in X^{**}. Also define

$$SEX(A) = \{x \in X: \hat{x} \text{ strongly exposes } A\}.$$

Here, a functional f *strongly exposes* a set B if the diameter of the *slice*

$$S(B, f, \alpha) = \{b \in B: f(b) \geqslant \alpha\}$$

approaches 0 as α approaches $\max f(B)$ from below.

In case $A \subset X^*$ is norm-closed we have $SEX(A) \subset SA(A)$. This inclusion is usually proper.

The present work was motivated by the following results.

THEOREM A. *The dual space X^* has the Radon-Nikodým property if and only if $SEX(A) \neq 0$ for all norm-closed, bounded, convex subsets, A of X^*.*

THEOREM B. *The dual space X^* has the Radon-Nikodým property if and only if $SA(A)$ is of 2nd category for all norm-closed, bounded, convex subsets A of X^*.*

The reader should refer to [1] or [2] for information on the Radon-Nikodým property (including its definition). Theorem A follows from a result of Namioka and Phelps [4] combined with one of Stegall [7]. Indeed, their results imply that, if X^* has the Radon-Nikodým property and A is a norm-closed, convex subset, then $SEX(A)$ is a dense G_δ subset of X. This fact also proves half of Theorem B. The other half is a very easy modification of an argument due to Bourgain (see [1], [6]).

Theorems A and B, together with a spirit of optimism, led to

Conjecture. If $SA(A) \neq \varnothing$ for all norm-closed, bounded, convex subsets A of X^*, then X^* has the Radon-Nikodým property.

Let us refer to the property of X which is conjectured to imply the *RNP* in X^* as *Property SA^**. Our result is weaker than the one conjectured. We were forced to

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assume that X is weakly compactly generated. This allows a reduction to the separable case. The desired conclusion is that every separable subspace of X has separable dual (that this implies that X^* has the *RNP* is due to Uhl [8] generalizing Dunford and Pettis [3]). We obtain the weaker conclusion that l^1 does not embed in X.

A lemma is needed.

LEMMA. $C[0, 1]$ *does not have property* SA^*.

PROOF. Let (x_n) be a dense sequence in $[0, 1]$. For each $n = 1, 2, \ldots,$ let μ_n be the point mass at x_n with total mass $n/(n + 1)$. Let

$$A = \overline{\mathrm{co}}\,((\mu_n) \cup (-\mu_n)) \subseteq C([0, 1])^*.$$

It is easy to see that

$$\sup\{\langle \mu, f\rangle : \mu \in A\} = \|f\|, \tag{1}$$

for any $f \in C([0, 1])$. Now we show that $\mu \in A$ then μ can be expressed as

$$\mu = \sum_{n=1}^{\infty} a_n \mu_n, \tag{2}$$

where $\sum_1^\infty |a_n| \leqslant 1$. To prove this, let T be the map from l^1 into $C([0, 1])^*$ which takes $(a_n) \in l^1$ to $\sum_1^\infty a_n\mu_n \in C([0, 1])^*$. Then T is clearly an isomorphism. Since A is obviously the image, under T, of the unit ball of l^1, (2) holds.

To finish the proof of the Lemma, we will show that no nonzero f in $C([0, 1])$ attains its supremum on A. Suppose the contrary. Then, using (1), there exists f in $C([0, 1])$ and $\mu \in A$ such that

$$\langle \mu, f\rangle = \|f\| > 0.$$

Expressing μ as in (2), we have

$$\begin{aligned} \|f\| = \langle \mu, f\rangle = \sum_1^\infty a_n \langle \mu_n, f\rangle &\leqslant \sum_1^\infty |a_n| |\langle \mu_n, f\rangle| \\ &< \sum_1^\infty |a_n|\, \|f\| \leqslant \|f\|. \end{aligned}$$

This contradiction completes the proof of the Lemma.

THEOREM. *Let X be a weakly compactly generated Banach space with property SA^*. Then l^1 does not embed in X.*

PROOF. We first observe that every quotient of a space with property SA^* also has this property. For, suppose Q is a bounded linear operator from a Banach space Y onto a Banach space Z. Suppose K is a closed, bounded, convex set in Z^* with $SA(K) = \varnothing$. A moments's reflection shows that $SA(Q^*K) = \varnothing$ and so Y fails to have SA^*.

Now suppose that l^1 embeds in X. Since X is weakly compactly generated there is a separable subspace Y of X which contains a copy of l^1 and which is complemented in X. Then Y also has property SA^*. But it follows from a result of

Pelczynski [5] that a separable space containing l^1 has $C[0, 1]$ as a quotient. Hence $C[0, 1]$ has SA^* and we have arrived at a contradiction. This completes the proof.

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References

1. R. Bourgin, *Notes on the Radon-Nikodym property* (to appear).

2. J. Diestel, *Geometry of Banach spaces*, Lecture Notes in Math., vol. 485, Springer-Verlag, New York and Berlin, 1975.

3. N. Dunford and B. J. Pettis, *Linear transformations on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323–392.

4. I. Namioka and R. R. Phelps, *Banach spaces which are Asplund spaces*, Duke Math. J. **42** (1975), 735–750.

5. A. Pelczynski, *On $C(S)$-subspaces of separable Banach spaces*, Studia Math. **31** (1968), 513–522.

6. J. Rainwater, Univ. of Washington Seminar Notes, Fall 1976.

7. C. Stegall, *The duality between Asplund spaces and spaces with the Radon-Nikodym property*, Israel J. Math. **29** (1978), 408–412.

8. J. J. Uhl, *A note on the Radon-Nikodym property for Banach spaces*, Rev. Roumaine Math. Pures Appl. **17** (1972), 113–115.

Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802