PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, November 1981

SOME INTRINSIC COORDINATES ON TEICHMÜLLER SPACE

CLIFFORD J. EARLE[1]

ABSTRACT. We give a new construction of intrinsic global coordinates on the Teichmüller space T_p of closed Riemann surfaces of genus $p > 2$. Our construction produces an injective holomorphic map from T_p into the space of Schottky groups of genus p.

1. Introduction. Since the Teichmüller space T_p of closed Riemann surfaces of genus $p \geqslant 2$ is a complex analytic manifold of dimension $n = 3p - 3$, any injective holomorphic map f: $T_p \to \mathbf{C}^n$ defines a set of global coordinate functions on T_p. We call these coordinates intrinsic if the coordinates $f(t)$ are determined from the marked Riemann surface t alone and do not depend on the choice of a basepoint t_0 in T_p. In this paper we describe a new way to define intrinsic coordinates on T_p.

We should emphasize that we are defining complex coordinates for the complex manifold T_p. The problem of finding real analytic coordinates was solved classically with the help of Fuchsian groups. The first global complex coordinates were found by Bers [2], using quasi-fuchsian groups. The Bers coordinates depend on the choice of a basepoint. Maskit [5] defined the first intrinsic (complex) coordinates. Our coordinates are closer in spirit to the Bers coordinates since we use quasi-fuchsian groups. It would, of course, be interesting to find global coordinates for T_p that do not depend on uniformization by Kleinian groups.

2. Quasifuchsian groups. Let Γ be a quasifuchsian group of type $(p, 0)$. This means that the limit set $\Lambda(\Gamma)$ is a Jordan curve in the extended plane, that Γ maps each of the Jordan regions D_1 and D_2 bounded by $\Lambda(\Gamma)$ into itself, and that the quotient maps $D_1 \to D_1/\Gamma$ and $D_2 \to D_2/\Gamma$ are unramified coverings of closed Riemann surfaces of genus p.

Lifting a canonical dissection of the surface D_1/Γ to D_1, we can choose an ordered $2p$-tuple

$$\sigma = (A_1, B_1, A_2, B_2, \ldots, A_p, B_p) \tag{1}$$

of Möbius transformations such that the A_j and B_j generate Γ and satisfy the relation

$$\prod_{j=1}^{p} C_j = I, \qquad C_j = A_j B_j A_j^{-1} B_j^{-1}. \tag{2}$$

Received by the editors August 10, 1980; presented to the Society, April 18, 1980.
1980 *Mathematics Subject Classification.* Primary 32G15; Secondary 30F40.
Key words and phrases. Riemann surface, Teichmüller space, quasifuchsian group.
[1]This research was partly supported by a grant from the National Science Foundation.

0002-9939/81/0000-0519/$02.25

The pair (σ, Γ) is called a marked quasifuchsian group. We say that (σ, Γ) is normalized if the attractive fixed points of B_1 and B_2 and the repulsive fixed point of B_1 are at 0, 1, and ∞ respectively. Our reason for that normalization will become clear in §5. It is well known (see [3]) that the space of normalized groups (σ, Γ) is a complex manifold, biholomorphically equivalent to $T_p \times T_p$.

To represent T_p by a set of normalized groups one must embed T_p in $T_p \times T_p$. The Bers coordinates are obtained by identifying T_p with a "slice" $T_p \times \{t_0\}$ in $T_p \times T_p$. Our method is to identify T_p with the diagonal. Our main theorem gives a general procedure for making that identification, and in §5 we illustrate how to use the main theorem to define intrinsic coordinates on T_p.

3. The main theorem. By definition, if (σ, Γ) is a marked quasifuchsian group the $2p$-tuple σ induces a canonical dissection of D_1/Γ. The induced dissection of D_2/Γ, however, is not canonical, because of orientation. To identify the space of normalized groups with $T_p \times T_p$ we use a sense-reversing diffeomorphism to make the dissection of D_2/Γ canonical. The following theorem describes the diagonal.

THEOREM 1. *Let W be a closed Riemann surface of genus $p \geq 2$ with a canonical homotopy basis $a_1, \ldots, b_p$, and let θ be an automorphism of $\pi_1(W)$ induced by a sense-reversing diffeomorphism of W. There is a unique normalized marked quasi-fuchsian group (σ, Γ) such that:*

(i) *the map from $\pi_1(W)$ to Γ that sends a_j to A_j and b_j to B_j, $1 \leq j \leq p$, is induced by a conformal map from W to D_1/Γ,*

(ii) *there is a conformal map F: $D_2 \to D_1$ such that*

$$F(\gamma z) = \theta(\gamma)F(z) \quad \text{for all } \gamma \in \Gamma, z \in D_2.$$

If θ has order two, then F is a Möbius transformation of order two, and F and Γ generate a Kleinian group whose deformation space is T_p.

Notice that in (ii) we use the isomorphism (i) between $\pi_1(W)$ and Γ to interpret θ as an automorphism of Γ. We refer the reader to [3] for a discussion of deformation spaces of Kleinian groups.

4. Proof of Theorem 1. This result is really a corollary of the simultaneous uniformization theorem of Bers [1], but we find it simplest to give a direct proof, modelled on the proof of [1]. First we choose a holomorphic universal covering of W by the upper half-plane U, identifying $\pi_1(W)$ with the group G of deck transformations in the usual way. We normalize G so that the attractive fixed points of b_1 and b_2 and the repulsive fixed point of b_1 are at 0, 1, and ∞ respectively. By hypothesis there is a sense-reversing diffeomorphism of W that induces the automorphism θ. Lifting to U we get a diffeomorphism f: $U \to U$ such that $f(gz) = \theta(g)f(z)$ for all $g \in G$, $z \in U$. Put $h(z) = f(\bar{z})$ for z in the lower half-plane U^*. Then h is a sense-preserving diffeomorphism of U^* onto U, and

$$h(gz) = \theta(g)h(z) \quad \text{for all } g \in G, z \in U^*. \tag{3}$$

Now let w: $\mathbf{C} \to \mathbf{C}$ be a quasiconformal map such that w fixes the points 0, 1, and ∞, and both w and $w \circ h^{-1}$ are conformal in U (i.e., $w_{\bar{z}} = 0$ in U and

$w_{\bar{z}}/w_z = h_{\bar{z}}/h_z$ in U^*). Put $\Gamma = wGw^{-1}$, define the isomorphism $\varphi: G \to \Gamma$ by

(4) $$\varphi(g) = wgw^{-1} \quad \text{for all } g \in G,$$

and put $\sigma = (A_1, B_1, \ldots, A_p, B_p)$, $A_j = \varphi(a_j)$, $B_j = \varphi(b_j)$, $1 \leqslant j \leqslant p$. Then (σ, Γ) is a normalized marked quasifuchsian group with $D_1 = w(U)$ and $D_2 = w(U^*)$, and the conformal map $w: U \to D_1$ induces a conformal map of W onto D_1/Γ that satisfies (i). Moreover $F = whw^{-1}: D_2 \to D_1$ is conformal, and (3) and (4) give

$$F\varphi(g)F^{-1} = whgh^{-1}w^{-1} = w\theta(g)w^{-1} = \varphi(\theta(g))$$

in D_1, so (σ, Γ) satisfies (i) and (ii).

Suppose (σ', Γ') is another normalized group that satisfies (i) and (ii) with F': $D_2' \to D_1'$ conformal. Write $\sigma' = (A_1', \ldots, B_p')$. Then (i) gives a conformal map H: $D_1 \to D_1'$, so that in D_1' we have

$$HA_jH^{-1} = A_j', \quad HB_jH^{-1} = B_j', \qquad 1 \leqslant j \leqslant p.$$

Put $C = H$ in D_1 and $C = (F')^{-1}HF$ in D_2. Then C maps the regular set of Γ conformally onto the regular set of Γ' and induces an isomorphism of Γ onto Γ', so the Marden isomorphism theorem [4] implies that C is a Möbius transformation. The normalization implies that C is the identity, so $(\sigma, \Gamma) = (\sigma', \Gamma')$ and (σ, Γ) is unique.

Finally, let θ have order two. Put $C = F$ in D_2 and $C = F^{-1}$ in D_1. Then $C\gamma C^{-1} = \theta(\gamma)$ in both D_1 and D_2, so the Marden isomorphism theorem again implies that C is a Möbius transformation. By construction C has order two. Since $F = C$ in D_2, F is itself (extendible to) a Möbius transformation of order two. It is clear from (ii) that the group H generated by Γ and F is Kleinian, and Γ is the subgroup of index two that maps the region D_1 onto itself. By §7 of [3], the deformation space of H is biholomorphically equivalent to T_p. The equivalence is obtained in the natural way. Each point in the deformation space determines a marked quasifuchsian subgroup (σ, Γ), which in turn determines a marked Riemann surface $(D_1/\Gamma, \sigma)$ in T_p. Theorem 1 is proved.

5. Intrinsic global coordinates. To use Theorem 1 we must choose an automorphism θ. Let (σ, Γ) be a marked quasifuchsian group with σ given by (1). Then Γ is the free group on $A_1, \ldots, B_p$, modulo the relation (2). Put

(5) $$K_0 = I, \quad K_j = \prod_{l=1}^{j} C_l, \qquad 1 \leqslant j \leqslant p,$$

and define θ on generators by

(6) $$\theta(A_j) = K_{j-1}A_j^{-1}K_{j-1}^{-1}, \quad \theta(B_j) = K_jB_jK_{j-1}^{-1}, \qquad 1 \leqslant j \leqslant p.$$

It is easy to prove by induction on j that

(7) $$\theta(K_j) = K_j^{-1}, \qquad 1 \leqslant j \leqslant p.$$

The case $j = p$ of (7) shows that θ preserves the relation (2) and does indeed define an automorphism of Γ. It is clear from (6) and (7) that θ has order two.

Every automorphism of Γ is induced by some diffeomorphism of D_1/Γ, and any diffeomorphism that induces θ is sense-reversing since on the level of homology θ fixes each B_j and reverses each A_j. We can therefore apply Theorem 1 to θ.

THEOREM 2. *If θ is defined by* (6), *the normalized group* (σ, Γ) *given by Theorem* 1 *is determined by $B_1, \ldots, B_p$. The multipliers of the B_j, the repulsive fixed points of $B_2, \ldots, B_p$, and the attractive fixed points of $B_3, \ldots, B_p$ are a global coordinate system for T_p.*

PROOF. Let (σ, Γ) be given by Theorem 1. Since θ has order two, there is a Möbius transformation F such that

$$F^2 = I \quad \text{and} \quad \theta(\gamma) = F\gamma F^{-1} \quad \text{for all } \gamma \in \Gamma. \tag{8}$$

Formula (6) implies by induction on j that

$$\begin{aligned} K_j &= \theta(B_j \cdots B_1)(B_j \cdots B_1)^{-1} \\ &= F(B_j \cdots B_1)F^{-1}(B_j \cdots B_1)^{-1}, \quad 1 \leqslant j \leqslant p. \end{aligned} \tag{9}$$

Taking $j = p$ in (9) we see that F is the unique Möbius transformation of order two that commutes with the loxodromic transformation $B_p \cdots B_1$. Thus the B_j determine F and, by (9), each K_j.

Now put

$$F_j = FK_{j-1}A_j, \qquad 1 \leqslant j \leqslant p. \tag{10}$$

We claim that F_j is the unique Möbius transformation of order two that commutes with B_j. First, F_j is not the identity map since it interchanges the regions D_1 and D_2. Next,

$$F_j^2 = FK_{j-1}A_jFK_{j-1}A_j = \theta(K_{j-1}A_j)K_{j-1}A_j = I,$$

by (6), (7), and (8). Finally,

$$\begin{aligned} F_jB_jF_j^{-1} &= F_jB_jF_j = \theta(K_{j-1}A_jB_j)K_{j-1}A_j \\ &= A_j^{-1}K_{j-1}^{-1}K_jB_jA_j = A_j^{-1}C_jB_jA_j = B_j, \end{aligned}$$

which proves our claim.

Since the B_j uniquely determine F and the F_j, formulas (9) and (10) show that the B_j determine (σ, Γ). That proves the first statement of Theorem 2. The second statement follows easily. Indeed, the fixed points and multipliers of the B_j are holomorphic functions on the deformation space of the Kleinian group generated by Γ and F (see §8 of [3]), hence on T_p. Since these fixed points and multipliers determine the B_j, and hence (σ, Γ), we have defined an injective holomorphic map from T_p into $\mathbf{C}^n$. The theorem is proved.

We remark in conclusion that $\{B_1, \ldots, B_p\}$ generates a Schottky group of genus p, and our coordinates on T_p give an injective holomorphic map of T_p into the space of Schottky groups of genus p. We will study the geometry of that map in a forthcoming paper.

References

1. L. Bers, *Simultaneous uniformization*, Bull. Amer. Math. Soc. **66** (1960), 94–97.

2. ______, *Correction to "Spaces of Riemann surfaces as bounded domains"*, Bull. Amer. Math. Soc. **67** (1961), 465–466.

3. I. Kra, *Deformation spaces*, '*A crash course on Kleinian groups*', Lecture Notes in Math., vol. 400, Springer-Verlag, Berlin and New York, 1974, pp. 48–70.

4. A. Marden, *Geometrically finite Kleinian groups and their deformation spaces*, '*Discrete Groups and Automorphic Functions*', Academic Press, London and New York, 1977, pp. 259–293.

5. B. Maskit, *Moduli of marked Riemann surfaces*, Bull. Amer. Math. Soc. **80** (1974), 773–777.

Department of Mathematics, Cornell University, Ithaca, New York 14853