PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, November 1981

ON A MEAN ERGODIC THEOREM

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ABSTRACT. A recent mean ergodic theorem of Shaw [2] is improved. The proof given below is simple and direct.

The purpose of this paper is to prove the following

THEOREM. *Let $(T_t\colon t > 0)$ be a strongly continuous semigroup of uniformly bounded linear operators on a Banach space X. Suppose there exists $\delta > 0$ such that $\|T_t - I\| < 2$ for all $0 < t < \delta$. Then for each $0 < t < \delta$*

$$\lim_{b\to\infty} \frac{1}{b}\int_0^b T_s x\, dx = \lim_n \frac{1}{n}\sum_{i=0}^{n-1} T_{it}x$$

provided one of these limits exists.

PROOF. If X_0 [resp. X_t, $t > 0$] is the set of all x in X at which

$$P_0 x := \lim_{b\to\infty} \frac{1}{b}\int_0^b T_s x\, ds \quad \left[\text{resp. } P_t x := \lim_n \frac{1}{n}\sum_{i=0}^{n-1} T_{it}x\right]$$

exists, then, as is well known (see e.g. [1, Corollaries VIII.5.2 and VIII.7.2]),

$$X_0 = \left[\bigcap_{s>0} N(T_s - I)\right] \oplus \overline{\left[\bigcup_{s>0} R(T_s - I)\right]}$$

and

$$X_t = N(T_t - I) \oplus \overline{R(T_t - I)}.$$

Moreover if $x \in N(T_t - I)$ (i.e. $T_t x = x$) then

$$\lim_{b\to\infty} \frac{1}{b}\int_0^b T_s x\, dx = \lim_n \frac{1}{nt}\int_0^{nt} T_s x\, ds = \frac{1}{t}\int_0^t T_s x\, ds,$$

and thus $x \in X_0$. Therefore $X_t \subset X_0$, and hence to prove the theorem it suffices to show that

$$\overline{\left[\bigcup_{s>0} R(T_s - I)\right]} \subset \overline{R(T_t - I)} \qquad (0 < t < \delta).$$

To do this, however, it also suffices to show that

$$\overline{R(T_{t/2} - I)} \subset \overline{R(T_t - I)} \qquad (0 < t < \delta),$$

because the semigroup $(T_t\colon t > 0)$ is strongly continuous.

Received by the editors September 20, 1980.
1980 *Mathematics Subject Classification.* Primary 47A35, 47D05.
Key words and phrases. Semigroups of operators, mean ergodic theorem.

Suppose $x \in \overline{R(T_{t/2} - I)}$, which is equivalent to $P_{t/2}x = 0$. Then

$$\frac{1}{2n}\sum_{i=0}^{2n-1} T_{it/2} = \frac{1}{2}\left[I + T_{t/2}\right]\left(\frac{1}{n}\sum_{i=0}^{n-1} T_{it}x\right)$$

$$= \left[I + \frac{1}{2}(T_{t/2} - I)\right]\left(\frac{1}{n}\sum_{i=0}^{n-1} T_{it}x\right)$$

and thus

$$\left\|\frac{1}{n}\sum_{i=0}^{n-1} T_{it}x\right\| \leqslant \left\|\frac{1}{2n}\sum_{i=0}^{2n-1} T_{it/2}x\right\| + \frac{1}{2}\left\|T_{t/2} - I\right\| \left\|\frac{1}{n}\sum_{i=0}^{n-1} T_{it}x\right\|,$$

therefore

$$\left\|\frac{1}{n}\sum_{i=0}^{n-1} T_{it}x\right\| \leqslant \left(1 - \frac{1}{2}\left\|T_{t/2} - I\right\|\right)^{-1}\left\|\frac{1}{2n}\sum_{i=0}^{2n-1} T_{it/2}x\right\| \to 0$$

as $n \to \infty$, since $1 - \frac{1}{2}\|T_{t/2} - I\| > 0$. This proves that $P_t x = 0$, or equivalently that $x \in \overline{R(T_t - I)}$. Hence the theorem is established.

References

1. N. Dunford and J. T. Schwartz, *Linear operators*. Part I: *General theory*, Interscience, New York, 1958.

2. S.-Y. Shaw, *Ergodic projections of continuous and discrete semigroups*, Proc. Amer. Math. Soc. **78** (1980), 69–76.

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