PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, November 1981

ON WEAKLY COMPACT OPERATORS ON BANACH LATTICES

C. D. ALIPRANTIS AND O. BURKINSHAW

ABSTRACT. Consider a Banach lattice E and an order bounded weakly compact operator $T: E \to E$. The purpose of this note is to study the weak compactness of operators that are related with T in some order sense. The main results are the following.

(1) If T is a positive weakly compact operator and an operator $S: E \to E$ satisfies $0 \leqslant S \leqslant T$, then S^2 is weakly compact. (Examples show that S need not be weakly compact.)

(2) If T and S are as in (1) and either S is an orthomorphism or E has an order continuous norm, then S is weakly compact.

(3) If E is an abstract L-space and T is weakly compact, then the modulus $|T|$ is weakly compact.

For notation and terminology concerning Banach lattices we follow [1] and [5]. Consider a Banach lattice E and a positive weakly compact operator $T: E \to E$. Now, if $S: E \to E$ is an operator such that $0 \leqslant S \leqslant T$ holds, then what effect does the weak compactness of T have on S? Before giving some positive answers to this question, we shall present two examples to show that (in general) under these conditions S need not be a weakly compact operator.

EXAMPLE 1. Let $\{r_n\}$ denote the sequence of Rademacher functions on [0, 1]. That is, $r_n(t) = \operatorname{Sgn} \sin(2^n \pi t)$. Consider $S, T: L_1[0, 1] \to l_\infty$ defined by

$$S(f) = \left(\int_0^1 f(x) r_n^+(x)\, dx\right) \quad \text{and} \quad T(f) = \left(\int_0^1 f(x)\, dx, \int_0^1 f(x)\, dx, \ldots\right).$$

Then T is compact (it has rank one), and $0 \leqslant S \leqslant T$ holds. On the other hand, S is not a weakly compact operator. To see this, start by observing that the sequence

$$u_n = (\underbrace{0, 0, \ldots, 0}_{n \text{ places}}, 1, 1, \ldots)$$

of l_∞ has no subsequence that converges weakly to zero. Indeed, if $\{w_n\}$ is a subsequence of $\{u_n\}$, then $\|w_n\|_\infty = 1$ for each n and $w_n \downarrow 0$ hold, which show that $\{w_n\}$ cannot converge weakly to zero [1, Theorem 9.8, p 63].

Now consider the sequence $\{f_k\}$ of $L_1[0, 1]$ defined by $f_k = 2^k \chi_{[0, 2^{-k}]}$. Clearly, $\|f_k\|_1 = 1$ holds for all k. An easy computation shows that

$$S(f_k) = (\underbrace{1, 1, \ldots, 1}_{k \text{ places}}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots).$$

Received by the editors September 3, 1980; presented to the Society, January 8, 1981.
1980 *Mathematics Subject Classification*. Primary 46A40, 46B30, 47B05, 47B55.
Key words and phrases. Banach lattice, positive operator, weakly compact operator.

0002-9939/81/0000-0530/$02.50

Also, it is easy to see that the only possible weak limit of any subsequence of $\{S(f_k)\}$ is $e = (1, 1, 1, \ldots)$. However, the relation $S(f_k) - e = -\frac{1}{2} u_k$ shows that no subsequence of $\{S(f_k)\}$ can converge weakly. In other words, S is not weakly compact. $\square$

EXAMPLE 2. Let S and T be as they were defined in Example 1. Consider the Banach lattice $E = L_1[0, 1] \oplus l_\infty$. Note that neither E nor E' has an order continuous norm. Now define $S_1, T_1\colon E \to E$ via the matrices

$$S_1 = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

Clearly, $0 \leqslant S_1 \leqslant T_1$ holds and T_1 is a compact operator. On the other hand, it should be obvious that S_1 is not weakly compact. $\square$

For a Banach lattice E we denote its norm dual by E' and its order continuous dual by $E_n^\sim$. If A is an ideal of E and B is an ideal of E', then the absolute weak topology $|\sigma|(B, A)$ is the locally convex-solid topology on B generated by the family of Riesz seminorms $\{\rho_x\colon x \in A\}$, where $\rho_x(f) = |f|(|x|)$ for each $f \in B$. For details see **[1]**.

In order to obtain our main result we need a lemma. As usual, E is identified with its canonical image in its double dual E''.

LEMMA 3. *If E is a Banach lattice, then E is $|\sigma|(E'', E')$-dense in the ideal generated by E in E''.*

PROOF. Let I be the ideal generated by E in E''. Clearly, $I \subseteq (E')_n^\sim$ holds. By **[1**, Theorem 19.10, p. 129] we know that E is $|\sigma|((E')_n^\sim, E')$-dense in $(E')_n^\sim$. Thus, in view of $E \subseteq I \subseteq E'' \cap (E')_n^\sim$ and the fact that $|\sigma|((E')_n^\sim, E')$ and $|\sigma|(E'', E')$ agree on I, it follows that E is $|\sigma|(E'', E')$-dense in I. $\square$

We now come to the main theorem of the paper. The result states that if a positive operator is dominated by a weakly compact operator, then its second power is always a weakly compact operator (although the operator need not be weakly compact).

THEOREM 4. *Let E be a Banach lattice and let $T\colon E \to E$ be a positive weakly compact operator. If an operator $S\colon E \to E$ satisfies $0 \leqslant S \leqslant T$, then S^2 is a weakly compact operator.*

PROOF. Let I be the ideal generated by E in E''. The weak compactness of T is equivalent to $T''(E'') \subseteq E$ **[3**, Theorem 2, p. 482]. Thus, if $0 \leqslant u \in E''$, then in view of $0 \leqslant S'' \leqslant T''$ we have $0 \leqslant S''(u) \leqslant T''(u) \in E$, so that

$$S''(E'') \subseteq I. \tag{1}$$

Next we claim that S'' also satisfies

$$S''(I) \subseteq E. \tag{2}$$

If this has been established, then (1) and (2) give $(S^2)''(E'') = S''[S''(E'')] \subseteq S''(I) \subseteq E$, i.e., that S^2 is weakly compact.

The rest of the proof is devoted to establishing (2). To this end, let $0 \leqslant u \in I$. Pick some $x \in E$ with $0 \leqslant u \leqslant x$. By Lemma 3, there exists a net $\{x_\alpha\}$ of E

converging to u for $|\sigma|(E'', E')$ (and hence, for $\sigma(E'', E')$) and satisfying $0 \leqslant x_\alpha \leqslant x$ for all α. Let B' denote the closed unit ball of E'. The weak compactness of T implies that $T'\colon (E', \sigma(E', E)) \to (E', \sigma(E', E''))$ is continuous [3, Lemma 7, p. 484]. Thus, $T'(B')$ is $\sigma(E', E'')$-compact, and therefore, $T'(B')$ is also compact for the coarser topology $\sigma(E', I)$.

Now let $\varepsilon > 0$. Combining Theorems 20.14 and 20.6 of [1] or by using [2, Theorem 3.2, p. 192] we see that there exists some $0 \leqslant g \in E'$ satisfying

$$\langle (|T'f| - g)^+, x\rangle < \varepsilon \quad \text{for all } f \in B'.$$

Choose α_1 such that $\langle g, |x_\alpha - x_\beta|\rangle < \varepsilon$ holds for all $\alpha, \beta \geqslant \alpha_1$. Then for $\alpha, \beta \geqslant \alpha_1$ and $f \in B'$ we have

$$\begin{aligned}
|\langle f, |Sx_\alpha - Sx_\beta|\rangle| &\leqslant \langle |f|, S|x_\alpha - x_\beta|\rangle \leqslant \langle |f|, T|x_\alpha - x_\beta|\rangle \\
&= \langle T'|f|, |x_\alpha - x_\beta|\rangle \leqslant \langle (T'|f| - g)^+, |x_\alpha - x_\beta|\rangle + \langle g, |x_\alpha - x_\beta|\rangle \\
&\leqslant 2\langle (T'|f| - g)^+, x\rangle + \langle g, |x_\alpha - x_\beta|\rangle < 3\varepsilon.
\end{aligned}$$

This implies easily $\|Sx_\alpha - Sx_\beta\| \leqslant 3\varepsilon$ for all $\alpha, \beta \geqslant \alpha_1$, so that $\{Sx_\alpha\}$ is a norm Cauchy net of E. If y is its norm limit in E, then using the weak* continuity of S'' we have the following $\sigma(E'', E')$-limits:

$$S''(u) = \lim S''x_\alpha = \lim Sx_\alpha = y \in E.$$

Thus, (2) is established, and the proof of the theorem is complete. □

The preceding proof (with minor adjustments) yields, in actuality, the following general version.

THEOREM 5. *Let F, G, and H be Banach lattices, $F \xrightarrow{S_1} G \xrightarrow{S_2} H$ and $F \xrightarrow{T_1} G \xrightarrow{T_2} H$ be positive operators such that $0 \leqslant S_i \leqslant T_i$ $(i = 1, 2)$ holds. If T_1 and T_2 are weakly compact operators, then the composition operator S_2S_1 is likewise a weakly compact operator.*

The proof of Theorem 5 can also be obtained directly from Theorem 4 as follows: Consider $E = F \oplus G \oplus H$, and $S, T\colon E \to E$ defined via the matrices

$$S = \begin{bmatrix} 0 & 0 & 0 \\ S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & 0 & 0 \\ T_1 & 0 & 0 \\ 0 & T_2 & 0 \end{bmatrix}.$$

Then $0 \leqslant S \leqslant T$ holds, T is weakly compact, and

$$S^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ S_2S_1 & 0 & 0 \end{bmatrix}.$$

By Theorem 4, S^2 is weakly compact, and so, S_2S_1 is likewise a weakly compact operator.

An immediate application of Theorem 4 is the following.

COROLLARY 6. *Let E be a Dedekind complete Banach lattice and $T\colon E \to E$ an order bounded operator. If the modulus $|T|$ of T is weakly compact, then T^2 is weakly compact.*

So far, it is clear from our discussion that if we wish to conclude from $0 \leqslant S \leqslant T$ (and the weak compactness of T) that S is weakly compact, then some assumptions must be added either to the space or to the operator. The next two results are of this type.

Recall that a Banach lattice is said to have an order continuous norm whenever $u_\alpha \downarrow 0$ implies $\|u_\alpha\| \downarrow 0$. Any Banach lattice with an order continuous norm is Dedekind complete; see **[1]** and **[5]**.

THEOREM 7. *Let E and F be two Banach lattices with either E' or F having an order continuous norm. Also assume that S, T: $E \to F$ are two operators satisfying $0 \leqslant S \leqslant T$. If T is weakly compact, then S is likewise weakly compact.*

PROOF. Assume first that F has an order continuous norm. Then F is an ideal of F''; see **[1**, pp. 60–61**]**. Thus, if $0 \leqslant u \in E''$, then the relation $0 \leqslant S''u \leqslant T''u \in F$ implies $S''u \in F$. That is, $S''(E'') \subseteq F$ holds, so that S is weakly compact.

Now assume that E' has an order continuous norm. In view of T': $F' \to E'$ and $0 \leqslant S' \leqslant T'$ the preceding case shows that S' is weakly compact. Therefore, by Gantmacher's theorem, S is likewise weakly compact, and the proof of the theorem is complete. $\square$

Note. Theorem 7 was also proved in **[8]** by a completely different method.

For the next theorem we need the notion of a positive orthomorphism. A positive operator T: $E \to E$ on a Riesz space E is said to be a *positive orthomorphism*, whenever $x \wedge y = 0$ implies $x \wedge Ty = 0$. If E is a Banach lattice, then a positive operator T: $E \to E$ is a positive orthomorphism if and only if there exists some $\alpha > 0$ such that $0 \leqslant T \leqslant \alpha I$ holds (where I now denotes the identity operator on E). See, for example, **[7]**.

THEOREM 8. *Let E be a Banach lattice and T: $E \to E$ a positive weakly compact operator. If a positive orthomorphism S: $E \to E$ satisfies $0 \leqslant S \leqslant T$, then S is weakly compact.*

PROOF. Assume first that E is Dedekind complete. Then by Freudenthal's spectral theorem **[4**, Theorem 40.2, p. 257**]**, S is the uniform limit (and hence, the norm limit) of a sequence each term of which is a finite sum of the form $\sum \alpha_i P_i$, where each P_i is a band projection, each $\alpha_i \geqslant 0$, and $0 \leqslant \sum \alpha_i P_i \leqslant S \leqslant T$ holds. In view of $0 \leqslant \alpha_i P_i \leqslant T$ and $P_i = P_i^2$, it follows from Theorem 4 that each $\sum \alpha_i P_i$ is a weakly compact operator. Thus, S is also a weakly compact operator.

The general case can be established by observing that $0 \leqslant S' \leqslant T'$ holds, S' is a positive orthomorphism on E', T' is weakly compact, and E' is Dedekind complete. $\square$

Now we turn our attention to the following problem. Consider two Banach lattices E and F with F Dedekind complete, and T: $E \to F$ an order bounded weakly compact operator.

Question. When is the modulus $|T|$ weakly compact?

Conditions under which $|T|$ is weakly compact are desirable. So far, we have been able to obtain an affirmative answer for two very special classes of Banach lattices.

Recall that a Banach lattice is said to have a Levi norm whenever it follows from $0 \leq x_\alpha \uparrow$ and $\sup\{\|x_\alpha\|\} < \infty$ that $\sup\{x_\alpha\}$ exists; see [1]. If the norm of a Banach lattice E is Levi and order continuous, then F is a band of F'' [1, Theorem 22.2, p. 159]. Thus, by [5, Theorem 1.5, p. 232] every continuous operator from an abstract L-space into a Banach lattice with a Levi order continuous norm is order bounded.

THEOREM 9. *Let E be an abstract L-space and F a Banach lattice with a Levi order continuous norm. If an operator T: $E \to F$ is weakly compact, then its modulus $|T|$ is also weakly compact.*

PROOF. Let T: $E \to F$ be weakly compact. We have already mentioned before that T must be order bounded, and so, $|T|$ exists.

Let B denote the closed unit ball of E. Then by our hypothesis, $T(B)$ is $\sigma(F, F')$-relatively compact. Also, from our assumptions it follows that $F = (F')_n^\sim$ [1, Theorem 22.2], and hence, $T(B)$ considered as a subset of $(F')_n^\sim$ is $\sigma((F')_n^\sim, F')$-relatively compact. Now combine Theorems 20.11 and 20.6 of [1] (or use [2, Theorem 2.14, p. 190]) to get that each order bounded disjoint sequence in F' converges to zero uniformly on the solid hull of $T(B)$. Next we claim that $|T|(B)$ also has this property. Indeed, if $0 \leq f \in F' \subseteq F_n^\sim$ and $x \in B$, then

$$
\begin{aligned}
|\langle f, |T|x\rangle| \leq \langle f, |T|(|x|)\rangle &= \sup\left\{\left\langle f, \sum_{i=1}^n |Tx_i|\right\rangle : x_i \geq 0 \text{ and } \sum_{i=1}^n x_i = |x|\right\} \\
&= \sup\left\{\sum_{i=1}^n \|x_i\|\left\langle f, \left|T\left(\frac{x_i}{\|x_i\|}\right)\right|\right\rangle : x_i \geq 0 \text{ and } \sum_{i=1}^n x_i = |x|\right\} \\
&\leq \sup\{\langle f, |Ty|\rangle : y \in B\} \cdot \sup\left\{\sum_{i=1}^n \|x_i\| : x_i \geq 0 \text{ and } \sum_{i=1}^n x_i = |x|\right\} \\
&\leq \sup\{\langle f, |Ty|\rangle : y \in B\}.
\end{aligned}
$$

This implies that each order bounded disjoint sequence in F' must also converge to zero uniformly on the solid hull of $|T|(B)$.

Now by [1, Theorem 20.6, p. 135], $|T|(B)$ is order-equicontinuous on F', and thus, by [1, Theorem 20.11, p. 139] and $F = (F')_n^\sim$, $|T|(B)$ is $\sigma(F, F')$-relatively compact. That is, $|T|$ is a weakly compact operator, and the proof is finished. $\square$

The preceding result coupled with Theorem 7 yields the following. ($\mathcal{L}_b(E, F)$ denotes the Riesz space of all order bounded operators from E into F.)

COROLLARY 10. *The weakly compact operators from an abstract L-space E into a Banach lattice F with a Levi order continuous norm form an ideal of $\mathcal{L}_b(E, F)$.*

When E is a L-space and F is a Banach function space on σ-finite measure spaces, Theorem 9 was proved in [6] by measure theoretical arguments.

Finally, we close the paper by mentioning that our results hold true for locally convex-solid Riesz spaces. Call a continuous operator T: $E \to F$ between two (Hausdorff) locally convex-solid Riesz spaces weakly compact if it carries topologically bounded subsets of E onto $\sigma(F, F')$-relatively compact subsets of F. For instance, a simple modification of the proof of Theorem 4 yields: *Let E be a* *topologically complete (Hausdorff) locally convex-solid Riesz space and let $T: E \to E$ be a positive weakly compact operator. If an operator $S: E \to E$ satisfies $0 \leq S \leq T$, then S^2 is a weakly compact operator.*

REFERENCES

1. C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, Academic Press, New York, 1978.

2. O. Burkinshaw, *Weak compactness in the order dual of a vector lattice*, Trans. Amer. Math. Soc. **187** (1974), 183–201.

3. N. Dunford and J. T. Schwartz, *Linear operators*. I, Wiley (Interscience), New York, 1958.

4. W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces*. I, North-Holland, Amsterdam, 1971.

5. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin and New York and Heidelberg, 1974.

6. A. R. Schep, *Compactness properties of an operator which imply that is is an integral operator*, Trans. Amer. Math. Soc. **265** (1981), 111–119.

7. A. W. Wickstead, *Representation and duality of multiplication operators on Archimedean Riesz spaces*, Compositio Math. **35** (1977), 225–238.

8. ______, *Extremal structure of cones of operators*, Quart. J. Math. Oxford Ser. (2) **32** (1981), 239–253.

DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY-PURDUE UNIVERSITY AT INDIANAPOLIS, INDIANAPOLIS, INDIANA 46205