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CURVATURE ESTIMATES FOR COMPLETE AND BOUNDED SUBMANIFOLDS IN A RIEMANNIAN MANIFOLD

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ABSTRACT. Let M be a complete n-dimensional submanifold in the $(2n-1)$-dimensional Euclidean space, with scalar curvature bounded from below. Baikousis and Koufogiorgos proved that the sectional curvature of M satisfies $\sup K_M \geq \lambda^{-2}$ if M is contained in a ball of radius λ. We extend this result to the case that the ambient space is a complete simply connected Riemannian manifold of nonpositive curvature.

1. Introduction. For $p < n$, let M be a complete n-dimensional Riemannian submanifold in the $(n+p)$-dimensional Euclidean space E^{n+p}. Under the assumption that the scalar curvature of M has a lower bound, Baikousis and Koufogiorgos [1] proved that if M is contained in a ball of radius λ, then the sectional curvature K_M of M satisfies $\sup K_M \geq \lambda^{-2}$. In this note we obtain a natural extension of the above inequality when the ambient space is a complete simply connected $(n+p)$-dimensional Riemannian manifold of nonpositive curvature. To state our result, we introduce a continuous function f: $[0, \infty) \to [1, \infty)$ by

$$f(t) = \begin{cases} 1 & \text{if } t = 0, \\ t\coth(t) & \text{if } t > 0. \end{cases} \tag{1}$$

THEOREM. *For $p < n$, let M be a complete n-dimensional Riemannian submanifold in a $(n+p)$-dimensional complete simply connected Riemannian manifold $\overline{M}$ whose sectional curvature satisfies $a \leq K_{\overline{M}} \leq b \leq 0$. If M is contained in a geodesic ball of radius λ and the scalar curvature of M has a lower bound, then the sectional curvature K_M of M satisfies* $\sup K_M \geq a + \lambda^{-2}\{f(\sqrt{-b}\,\lambda)\}^2$.

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2. Proof of Theorem. We denote the Riemannian metric on $\overline{M}$ (resp. M) by $\overline{\langle\ ,\ \rangle}$ (resp. $\langle\ ,\ \rangle$), the Riemannian connection by $\overline{\nabla}$ (resp. ∇), the Riemannian curvature tensor by $\overline{R}$ (resp. R) and the second fundamental form with respect to the immersion $M \subset \overline{M}$ by α.

Since the scalar curvature of M has a lower bound, we may assume $\inf K_M > -\infty$. Let d be the distance function on $\overline{M}$ and choose a point $\bar{o} \in \overline{M}$ such that $d(\bar{o}, x) \leq \lambda$ for all $x \in M$. We define a smooth function F: $M \to R$ by $F(x) = \{d(\bar{o}, x)\}^2/2$. Then by [4, Theorem A′] there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in M such that

$$\|\operatorname{grad} F(x_k)\| < k^{-1}, \tag{2}$$

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$$\nabla^2 F(X, X) < k^{-1} \quad \text{for all unit vectors } X \in T_{x_k}M, \tag{3}$$

$$\lim_{k\to\infty} F(x_k) = \sup F, \tag{4}$$

where $\nabla^2 F$ denotes the Hessian of F with respect to the Riemannian metric on M.

LEMMA 1. *Let* γ: $[0, 1] \to \overline{M}$ *be a geodesic in* $\overline{M}$ *such that* $\gamma(0) = \bar{o}$ *and* $\gamma(1) \in M$. *Then*

$$\nabla^2 F(X, X) \geqslant \langle \alpha(X, X), \dot\gamma(1)\rangle + L^{-2}\langle X, \dot\gamma(1)\rangle^2 + (\|X\|^2 - L^{-2}\langle X, \dot\gamma(1)\rangle^2) f(\sqrt{-b}\, L),$$

for all vectors X *tangent to* M *at* $\gamma(1)$, *where* L *is the length of* γ.

PROOF. Let $c(s)$ be the geodesic in M such that $\dot c(0) = X$ and let γ_s: $[0, 1] \to \overline{M}$ be the geodesic such that $\gamma_s(0) = \bar{0}$ and $\gamma_s(1) = c(s)$. Then we have $\nabla^2 F(X, X) = F(c(s))''|_{s=0} = E(\gamma_s)''|_{s=0}$, where $E(\gamma_s)$ is the energy of γ_s defined by $E(\gamma_s) = \int_0^1 \langle \dot\gamma_s, \dot\gamma_s\rangle/2$. Let V be the variation vector field along γ with respect to the variation $\{\gamma_s\}$. Then a calculation shows that

$$E(\gamma_s)''|_{s=0} = \langle \alpha(X, X), \dot\gamma(1)\rangle + I(V, V),$$

where $I(V, V) = \int_0^1 \{\langle \overline\nabla_{\dot\gamma} V, \overline\nabla_{\dot\gamma} V\rangle + \langle \overline{R}(\dot\gamma, V)\dot\gamma, V\rangle\}$. Let $\tilde M$ be the $(n + p)$-dimensional space form with constant curvature b and let σ: $[0, 1] \to \tilde M$ be a geodesic with length L. We construct a vector field W along σ such that $\|V\| = \|W\|$, $\|\overline\nabla_{\dot\gamma} V\| = \|\tilde\nabla_{\dot\sigma} W\|$ and $\langle V, \dot\gamma\rangle = \langle W, \dot\sigma\rangle$, where $\tilde\nabla$ is the Riemannian connection with respect to the Riemannian metric $\langle\ ,\ \rangle$ on $\tilde M$. Then $K_{\overline{M}} \leqslant b$ implies $I(V, V) \geqslant I(W, W)$. Let J be the Jacobi field along σ determined by $J(0) = 0$ and $J(1) = W(1)$. Then [2, First lemma, p. 24] implies $I(W, W) \geqslant I(J, J)$. Let U be the parallel vector field along σ determined by $U(1) = J(1) - L^{-2}\langle J(1), \dot\sigma(1)\rangle\dot\sigma(1)$, and let g: $[0, 1] \to R$ be the solution of $g'' + bL^2 g = 0$ determined by $g(0) = 0$ and $g(1) = 1$. Then we have $J(t) = g(t)U(t) + \{L^{-2}\langle J(1), \dot\sigma(1)\rangle t\}\dot\sigma(t)$ and $g'(1) = f(\sqrt{-b}\, L)$. Hence we see that $I(J, J) = \langle \tilde\nabla_{\dot\sigma} J, J\rangle|_{t=1} = g'(1)\|U(1)\|^2 + L^{-2}\langle J(1), \dot\sigma(1)\rangle^2 = f(\sqrt{-b}\, L)(\|X\|^2 - L^{-2}\langle X, \dot\gamma(1)\rangle^2) + L^{-2}\langle X, \dot\gamma(1)\rangle^2$. Q.E.D.

Let γ_k: $[0, 1] \to \overline{M}$ be the geodesic such that $\gamma_k(0) = \bar{o}$ and $\gamma_k(1) = x_k$, and let λ_k be the length of γ_k. We set $\lambda_\infty = \sup\{d(\bar{o}, x) | x \in M\}$, then (4) implies $\lim_{k\to\infty} \lambda_k = \lambda_\infty > 0$. Therefore we may assume $\lambda_k > 0$ for all k. Let X be a unit vector in $T_{x_k}M$. Then by (3) and Lemma 1 we have

$$k^{-1} > \langle \alpha(X, X), \dot\gamma_k(1)\rangle - \lambda_k^{-2}\langle X, \dot\gamma_k(1)\rangle^2\{f(\sqrt{-b}\,\lambda_k) - 1\} + f(\sqrt{-b}\,\lambda_k).$$

Since $\langle X, \dot\gamma_k(1)\rangle = \langle X, \operatorname{grad} F(x_k)\rangle$, (2) implies $\langle X, \dot\gamma_k(1)\rangle^2 < k^{-2}$. Hence we have

$$\|\alpha(X, X)\| > \{f(\sqrt{-b}\,\lambda_k) - A_k\}/\lambda_k \tag{5}$$

for all unit vectors $X \in T_{x_k}M$, where $A_k = k^{-1} + k^{-2}\lambda_k^{-2}\{f(\sqrt{-b}\,\lambda_k) - 1\}$. Since $\lim_{k\to\infty}\{f(\sqrt{-b}\,\lambda_k) - A_k\} = f(\sqrt{-b}\,\lambda_\infty) \geqslant 1$, we may assume $f(\sqrt{-b}\,\lambda_k) - A_k > 0$ for all k. Hence (5) implies $\alpha(X, X) \neq 0$ for all nonzero vectors $X \in T_{x_k}M$. Now we recall the following lemma [3, p. 28].

LEMMA 2. *Let* α: $R^n \times R^n \to R^p$ *be symmetric bilinear and satisfy* $\alpha(X, X) \neq 0$ *for all nonzero* $X \in R^n$. *If* $p < n$, *there exist linearly independent vectors* $X, Y \in R^n$ *such that* $\alpha(X, Y) = 0$, $\alpha(X, X) = \alpha(Y, Y)$.

By Lemma 2 there exist linearly independent vectors X_k, Y_k in $T_{x_k}M$ such that $\alpha(X_k, Y_k) = 0$, $\alpha(X_k, X_k) = \alpha(Y_k, Y_k)$. Hence by the Gauss equation, we have $\langle R(X_k, Y_k)Y_k, X_k\rangle = \langle \bar{R}(X_k, Y_k)Y_k, X_k\rangle + \|\alpha(X_k, X_k)\| \cdot \|\alpha(Y_k, Y_k)\|$. Let $\bar{K}(X_k, Y_k)$ (resp. $K(X_k, Y_k)$) be the sectional curvature of $\bar{M}$ (resp. M) for the plane spanned by X_k and Y_k. Then by (5) we see that

$$\begin{aligned}
K(X_k, Y_k) &= \bar{K}(X_k, Y_k) + \|\alpha(X_k, X_k)\| \\
&\quad \cdot \|\alpha(Y_k, Y_k)\| \left(\|X_k\|^2 \|Y_k\|^2 - \langle X_k, Y_k\rangle^2\right)^{-1} \\
&\geqslant a + \|\alpha(X_k, X_k)\| \cdot \|\alpha(Y_k, Y_k)\| \cdot \|X_k\|^{-2} \|Y_k\|^{-2} \\
&> a + \lambda_k^{-2}\left\{f\left(\sqrt{-b}\,\lambda_k\right) - A_k\right\}^2.
\end{aligned}$$

Letting k go to infinity, we have $\sup K_M \geqslant a + \lambda_\infty^{-2}\{f(\sqrt{-b}\,\lambda_\infty)\}^2$. Since $\lambda_\infty \leqslant \lambda$ and the function $t \mapsto t^{-2}\{f(\sqrt{-b}\,t)\}^2$ is decreasing, we have $\sup K_M \geqslant a + \lambda^{-2}\{f(\sqrt{-b}\,\lambda)\}^2$. This completes the proof of the theorem.

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