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SELECTIONS AND ORDERABILITY

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ABSTRACT. Let X be a compact Hausdorff space. Then X has a selection if and only if X is orderable.

0. Introduction. Let X be a compact Hausdorff space and let 2^X denote the hyperspace of nonempty closed subsets of X. A *selection* for X is a continuous map F: $2^X \to X$ such that $F(A) \in A$ for all $A \in 2^X$. Let $X(2)$ denote the 2-fold symmetric product of X, i.e. the subspace of 2^X consisting of all nonempty closed subspaces of X containing at most two points. A *weak selection* for X is a continuous map s: $X(2) \to X$ such that $s(A) \in A$ for all $A \in X(2)$. It is easy to see that X has a weak selection if and only if there is a continuous map s: $X^2 \to X$ such that for all $x, y \in X$,

(1) $s(x, y) = s(y, x)$, and

(2) $s(x, y) \in \{x, y\}$.

Such a map s: $X^2 \to X$ will also be called a weak selection.

Michael [M] showed that for a continuum X the following statements are equivalent: (a) X has a selection, (b) X has a weak selection, and (c) X is orderable. In [Y], Young claims, without giving a proof, that statements (a), (b), and (c) are also equivalent for compact zero-dimensional spaces X. In this paper we will show that, for compacta, statements (a), (b), and (c) are always equivalent.

1. The construction. Let X be compact and let s: $X^2 \to X$ be a weak selection. For each $x \in X$ define

$$B_x = \{y \in X \mid s(y, x) = y\},$$

and

$$A_x = \{y \in X \mid s(y, x) = x\}.$$

Observe that both A_x and B_x are closed, that $A_x \cup B_x = X$ and that $A_x \cap B_x = \{x\}$.

1.1. THEOREM. *Let X be a compact space. Then the following statements are equivalent*:

(a) *X is orderable*,

(b) *X has a weak selection*,

(c) *X has a selection.*

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PROOF. The implication (c) $\Rightarrow$ (b) is trivial and the implication (a) $\Rightarrow$ (c) is well known. Indeed, simply define F: $2^X \to X$ by $F(A) = \min(A)$. An easy check shows that F is a selection. It therefore suffices to prove that (b) $\Rightarrow$ (a). To this end, let s: $X^2 \to X$ be a weak selection for X and, for each $x \in X$, let A_x and B_x be defined as above. Let $\prec$ be a wellordering on X. For every $x \in X$ we will construct closed sets L_x, $U_x \subset X$ such that

(1) $L_x \cup U_x = X$ and $L_x \cap U_x = \{x\}$,

(2) if $y \prec x$ and if $x \in L_y$ then $L_x \subset L_y \setminus \{y\}$,

(3) if $y \prec x$ and if $x \in U_y$ then $U_x \subset U_y \setminus \{y\}$,

(4) if $z \in L_x$ and if $z \notin \bigcup \{L_y \mid y \prec x \,\&\, x \in U_y\}$ then $z \in B_x$,

(5) if $z \in U_x$ and if $z \notin \bigcup \{U_y \mid y \prec x \,\&\, x \in L_y\}$ then $z \in A_x$.

(In the total ordering on X which we will construct in this proof, L_x will be the set of all points smaller than or equal to x, and U_x will be the set of all points larger than or equal to x.)

Let x_0 be the first element of X and define $L_{x_0} = B_{x_0}$ and $U_{x_0} = A_{x_0}$. Assume that we have defined L_y and U_y for all $y \prec x$ satisfying (1) through (5). Let $E = \{y \prec x \mid x \notin L_y\}$ and $F = \{y \prec x \mid x \notin U_y\}$. Put

$$Z = X \setminus \Big(\bigcup_{y \in E} L_y \cup \bigcup_{y \in F} U_y \Big).$$

Let $\kappa = |E|$ and for each $\xi \leqslant \kappa$ define points $y_\xi \in E$ in the following way:

(6) $y_0 = \min(E)$,

(7) $y_\zeta = \min[\{x\} \cup \{y \in E \mid (y_\mu \prec y \text{ for all } \mu < \zeta) \,\&\, (y \notin \bigcup_{\mu<\zeta} L_{y_\mu})\}]$. Let $\xi \leqslant \kappa$ be the first ordinal for which $y_\xi = x$.

Claim 1. If $\xi_0 \leqslant \xi$ then $\bigcup \{L_y \mid y \in E \,\&\, y \prec y_{\xi_0}\} = \bigcup_{\mu<\xi_0} L_{y_\mu}$.

Take $y \in \{z \in E \mid z \prec y_{\xi_0}\} \setminus \{y_\mu \mid \mu < \xi_0\}$ and let $\mu \leqslant \xi_0$ be the first ordinal for which $y \prec y_\mu$. Since $y_\rho \prec y$ for all $\rho < \mu$ (notice that $\mu \neq 0$) and since $y \neq y_\mu$, by (7), $y \in \bigcup_{\rho<\mu} L_{y_\rho}$. Choose $\rho < \mu$ such that $y \in L_{y_\rho}$. Since $y_\rho \prec y$, by (2),

$$L_y \subset L_{y_\rho} \subset \bigcup_{\delta<\xi_0} L_{y_\delta}.$$

Claim 2. If $\mu_0 < \mu_1 < \xi$ then $L_{y_{\mu_0}} \subset L_{y_{\mu_1}} \setminus \{y_{\mu_1}\}$.

By (7), $y_{\mu_1} \notin L_{y_{\mu_0}}$. Consequently, $y_{\mu_1} \in U_{y_{\mu_0}}$ and therefore, by (3), $U_{y_{\mu_1}} \subset U_{y_{\mu_0}} \setminus \{y_{\mu_0}\}$. Consequently, by (1), $L_{y_{\mu_0}} \subset L_{y_{\mu_1}} \setminus \{y_{\mu_1}\}$.

Claim 3. If $\mu_0 < \mu_1 < \xi$ then $L_{y_{\mu_1}} \setminus L_{y_{\mu_0}} \subset A_{y_{\mu_0}}$.

Take $t \in L_{y_{\mu_1}} \setminus L_{y_{\mu_0}}$. Since $t \in U_{y_{\mu_0}}$ and, by (5),

$$U_{y_{\mu_0}} \subset \bigcup \{U_y \mid y \prec y_{\mu_0} \,\&\, y_{\mu_0} \in L_y\} \cup A_{y_{\mu_0}},$$

we may assume, without loss of generality that $t \in U_z$ for certain $z \prec y_{\mu_0}$ with $y_{\mu_0} \in L_z$; we will reach a contradiction. Assume that $y_{\mu_1} \in L_z$. Since $y_{\mu_0} \prec y_{\mu_1}$ and since $z \prec y_{\mu_0}$ this implies by (2), that $L_{y_{\mu_1}} \subset L_z \setminus \{z\}$. Consequently, $t \in L_z \setminus \{z\}$ and $t \in U_z$, contradicting (1). This shows that $y_{\mu_1} \notin L_z$ which implies that $y_{\mu_1} \in U_z$. Since $z \prec y_{\mu_1}$, by (3), $U_{y_{\mu_1}} \subset U_z$ and therefore $x \in U_z$. If also $x \in L_z$ then $x = z$ which is impossible since $z \prec x$. We conclude that $x \notin L_z$ or equivalently,

$z \in E$. Let $\varepsilon \leqslant \mu_0$ be the smallest ordinal such that $z \preccurlyeq y_\varepsilon$. Since $y_\delta \prec z$ for every $\delta < \varepsilon$ by (7), either $z = y_\varepsilon$ or $z \in l_{y_\delta}$ for certain $\delta < \varepsilon$. If $z = y_\varepsilon$ then $y_{\mu_0} \in L_{y_\varepsilon}$ which contradicts $z \prec y_{\mu_0}$ (Claim 2). Therefore, $z \in L_{y_\delta}$ for certain $\delta < \varepsilon$. Then $z \in L_{y_\delta} \subset L_{y_{\mu_0}} \setminus \{y_{\mu_0}\}$. Since $z \prec y_{\mu_0}$ and since $y_{\mu_0} \in L_z$, by (2), we also have that

$$L_{y_{\mu_0}} \subset L_z \setminus \{z\},$$

which implies that $z \in L_{y_{\mu_0}} \subset L_z \setminus \{z\}$, a contradiction.

Claim 4. If $t \in \mathrm{Cl}_X(\bigcup_{y \in E} L_y) \setminus \bigcup_{y \in E} L_y$ then t is a cluster point of the net $\{y_\mu \mid \mu < \xi\}$.

Suppose not and take a closed neighborhood C of t which misses

$$\mathrm{Cl}_X\{y_\mu \mid \mu < \xi\}.$$

From Claim 1 it is clear that there is a cofinal subset $G \subset \xi$ with the property that for each $\mu \in G$ there exists a point $c_\mu \in C \cap L_{y_\mu}$ such that

$$\mu = \min\{\delta < \xi \mid c_\mu \in L_{y_\delta}\}.$$

Take $\mu \in G$. We claim that $c_\mu \in B_{y_\mu}$. If not, then by (4) there is a $y \prec y_\mu$ such that $c_\mu \in L_y$ and $y_\mu \in U_y$. Since $y \prec y_\mu$ and $y_\mu \in U_y$, by (3), $U_{y_\mu} \subset U_y \setminus \{y\}$ which implies that $L_y \subset L_{y_\mu}$. Consequently, $x \notin L_y$, since $x \notin L_{y_\mu}$, or equivalently, $y \in E$. By Claim 1 we can find $\delta < \mu$ such that $c_\mu \in L_{y_\delta}$, which is a contradiction since $\mu = \min\{\delta < \xi \mid c_\mu \in L_{y_\delta}\}$. This implies that for all $\mu \in G$ we have that $s(c_\mu, y_\mu) = c_\mu$.

Let (c, y) be a cluster point of the net $\{(c_\mu, y_\mu)\}_{\mu \in G}$. Then $c \in C$ and $y \notin C$, and since $s(c_\mu, y_\mu) = c_\mu \in C$ for all $\mu \in G$ it is clear that $s(c, y) = c$. Next take $\mu \in G$ arbitrarily. For all $\delta > \mu$ we have by Claim 3 that $s(y_\mu, c_\delta) = y_\mu$. Hence $s(c, y_\mu) = s(y_\mu, c) = y_\mu$. This would imply that $s(c, y) = y$, and since $y \neq c$ this is a contradiction.

Claim 5. If both t and u are cluster points of the net $\{y_\mu \mid \mu < \xi\}$ then $t = u$.

Let C and D be closed and disjoint neighborhoods of, respectively, t and u. There is clearly a cofinal subset $G \subset \xi$ and for each $\mu \in G$ points

$$c_\mu \in C \cap \{y_\lambda \mid \lambda < \xi\} \quad \text{and} \quad d_\mu \in D \cap \{y_\lambda \mid \lambda < \xi\}$$

such that if $\mu, \delta \in G$ and $\mu < \delta$ then

$$c_\mu \prec d_\mu \prec c_\delta.$$

Let (t', u') be a cluster point of the net $\{(c_\mu, d_\mu)\}_{\mu \in G}$, then $t' \in C$ and $u' \in D$. By Claim 3, $s(c_\mu, d_\mu) = c_\mu$ and consequently, $s(u', t') = t'$. Fix $\mu \in G$. For each $\delta > \mu$ it is clear that $s(d_\mu, c_\delta) = d_\mu$ (Claim 3). Since $t' \in \mathrm{Cl}_X\{c_\delta \mid \delta > \mu\}$ this implies that

$$s(d_\mu, t') = d_\mu.$$

Since $(u', t') \in \mathrm{Cl}_{X^2}\{(d_\mu, t') \mid \mu \in G\}$ this implies that $s(u', t') = u'$. Since $u' \neq t'$, this is a contradiction.

Claim 6. $\bigcup_{y \in E} L_y$ has at most one boundary point.

Follows immediately from Claims 4 and 5.

Claim 7. If $t \in Z$ and $\mu < \xi$ then $t \in A_{y_\mu}$.

Since $t \notin L_{y_\mu}$ clearly $t \in U_{y_\mu}$. T. erefore by (5), if $t \notin A_{y_\mu}$ then $t \in U_y$ for certain $y \prec y_\mu$ with $y_\mu \in L_y$. If $x \in L_y$ then $x \notin U_y$ since $x \neq y$ in which case $Z \cap U_y = \varnothing$ which contradicts $t \in Z \cap U_y$. Therefore $y \in E$. By Claim 1

$$\bigcup \{L_y \mid y \in E \,\&\, y \prec y_\mu\} = \bigcup_{\delta<\mu} L_{y_\delta}.$$

Therefore $y_\mu \in L_{y_\delta}$ for certain $\delta < \mu$ which contradicts (7).

Formally we have to consider two cases, namely that ξ is a successor or that ξ is a limit ordinal. Those two cases can be treated analogously and since the case that ξ is a limit is more complicated we will assume from now on that ξ is a limit.

Since $L_{y_\mu} \setminus \{y_\mu\}$ is open for each $\mu < \xi$, by Claims 1 and 2, $\bigcup_{y\in E} L_y$ must have a limit point, say a, and by Claim 6 we see that a is unique. By using precisely the same technique as above and again restricting our attention to the limit case we can find a limit ordinal η and for each $\mu < \eta$ a point $z_\mu \in F$ such that

(8) if $\mu < \delta$ then $U_{z_\mu} \subset U_{z_\delta}$,

(9) $\bigcup_{\mu<\eta} U_{z_\mu} = \bigcup_{y\in F} U_y$, and

(10) if $t \in Z$ and $\mu < \eta$ then $t \in B_{z_\mu}$.

Again we find that $\bigcup_{y\in F} U_y$ has a unique boundary point, say b, and that this point is a cluster point of the net $\{z_\mu \mid \mu < \eta\}$.

(Note that, by (1), (2) and (3), $y \in E$ and $y' \in F$ implies that $L_y \cap U_{y'} = \varnothing$.)

Case 1. $a = b$. We then claim that $Z = \{x\} = \{a\} = \{b\}$. For assume that $t \in Z$. By Claim 7, $s(y_\mu, t) = y_\mu$ for all $\mu < \xi$ and consequently $s(a, t) = a$ since a is a limit point of $\{y_\mu\}_{\mu<\xi}$. On the other hand, by (10), $s(t, z_\mu) = t$ for all $\mu < \eta$. By the same argument $s(t, a) = s(t, b) = t$. Hence $t = a$.

We therefore conclude that $a = b = x$ and that $Z = \{x\}$. Now define

$$L_x = \bigcup_{y\in E} L_y \cup \{x\} \quad \text{and} \quad U_x = \bigcup_{y\in F} U_y \cup \{x\}.$$

An easy check shows that our inductive hypotheses are satisfied.

Case 2. $a \neq b$ and $x \notin \{a, b\}$. Define $L_x = \bigcup_{y\in E} L_y \cup (Z \cap B_x)$ and $U_x = \bigcup_{y\in F} U_y \cup (Z \cap A_x)$. Observe that both L_x and U_x are closed since $a \in Z \cap B_x$ and $b \in Z \cap A_x$. Again an easy check shows that our inductive hypotheses are satisfied.

Case 3. $x = a$ and $a \neq b$. Define $L_x = \bigcup_{y\in E} L_y \cup \{x\}$ and $U_x = \bigcap_{\mu<\xi} U_{y_\mu}$.

Case 4. $x = b$ and $a \neq b$. Similar to Case 3.

Now define $x \leqslant y$ iff $x \in L_y$. Then $\leqslant$ is a linear order which generates the topology of X since X is compact and since for each $x \in X$ the sets $\{y \in X \mid y \leqslant x\}$ and $\{y \in X \mid x \leqslant y\}$ are closed.

2. Notes. A space X is called weakly orderable (abbreviated KOTS) provided that there is a linear order $\leqslant$ on X such that for each $y \in X$ the sets $\{x \in X \mid x \leqslant y\}$ and $\{x \in X \mid y \leqslant x\}$ are both closed. It is easily seen that whenever X is a KOTS then the function $s\colon X^2 \to X$ defined by $s(x,y) = \min\{x,y\}$ is a weak selection. This suggests the following question:

Question. Let X be a space. Is X a KOTS if and only if X admits a weak selection?

The technique used in the proof of our theorem is not applicable to answer this question since certain transfinite sequences of points need not have limit points.

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