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ON THE SEGAL CONJECTURE FOR $Z_2 \times Z_2$

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ABSTRACT. The Segal conjecture regarding the Burnside ring and stable cohomotopy of a finite group G is reduced for the case $G = Z_2 \times Z_2$ to a statement about Ext groups. This statement has since been proved by H. Miller, J. F. Adams and J. H. C. Gonawardena.

The Segal conjecture states that for any finite group G, there is an isomorphism $\hat{\alpha}_G$: $A(G)\hat{\to} \pi_s^0(BG)$ from the completed Burnside ring to the zeroth stable cohomotopy group of its classifying space. The conjecture was proved for cyclic groups in [**4**, **2**, and **9**]. In this paper we reduce the conjecture for $Z_2 \times Z_2$ to a statement, (1), about Ext groups. This Ext statement was conjectured by Davis in [**1**] based upon extensive calculations and is proved by Adams, Gunawardena, and Miller in [**7**].

Let $P = Z_2[x, x^{-1}]$ be made into a module over the mod 2 Steenrod algebra A as in [**5**], and for $-\infty \leqslant k \leqslant n \leqslant \infty$ let P_k^n be the subquotient of P which is nonzero in degree k through n, inclusive. If $n = \infty$ or $k = -\infty$, they will usually be omitted from the notation. The suspension $\Sigma^j M$ of a graded module M is defined by $(\Sigma^j M)_{i+j} = M_i$.

STATEMENT 1. There is an epimorphism of A-modules

$$\Sigma P \otimes \Sigma P \xrightarrow{\phi} Z_2 \oplus \Sigma P_0$$

which induces an isomorphism in $\mathrm{Ext}_A(\,, Z_2)$. In fact

$$\phi(sx^a \otimes sx^b) = \left(\binom{0}{a+1}\binom{0}{b+1}, \binom{a}{a+b+1} sx^{a+b+1}\right).$$

As mentioned above, Statement 1 is proved in [**7**]. The main result of this paper is

THEOREM 2. *Statement* 1 *implies that* $\hat{\alpha}_{Z_2 \times Z_2}$ *is an isomorphism.*

The proof of Theorem 2 mimicks [**4**]. The main part is to use Statement 1 to calculate $\pi_s^0(RP^\infty \times RP^\infty)$ via the Adams spectral sequence. We begin by deducing from (1) the Ext groups relevant to $[RP_1^\infty \wedge RP_1^\infty, S^0]$, the group of stable homotopy classes of maps.

DEFINITION 3. If M is a (left) A-module, let DM denote the dual module, made into a left A-module using the antiautomorphism χ; i.e., $(DM)_k = \mathrm{Hom}_{Z_2}(M_{-k}, Z_2)$ with $(\theta \cdot \phi)(m) = \phi(\chi(\theta)m)$.

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PROPOSITION 4. (i) $D(P_k^n) \approx \Sigma P_{-n-1}^{-k-1}$ *if* $-\infty < k < n < \infty$.
(ii) *If M has finite type*, $\mathrm{Ext}_A^{s,t}(L, M) \approx \mathrm{Ext}_A^{s,t}(DM, DL)$.

PROOF. (i) is well known. If N is a left A-module, let N_R denote the associated right A-module defined using χ. Let M^* denote the dual as in [5, 4.3]. Then $(N_R)^* = DN$. (ii) follows from [5, 4.3] and the symmetry of Tor as follows.

$$\begin{aligned}\mathrm{Ext}(L, M) &\approx \mathrm{Ext}(L, M^{**}) \approx (\mathrm{Tor}(M^*, L))^* \\ &\approx (\mathrm{Tor}(L_R, DM))^* \approx \mathrm{Ext}(DM, DL). \quad \square\end{aligned}$$

COROLLARY 5. $\mathrm{Ext}_A^{s,t}(Z_2, P_1 \otimes P_1) \approx \mathrm{Ext}_A^{s,t}(\Sigma^2 P^{-2} \otimes P^{-2}, Z_2)$.

We will also need

PROPOSITION 6. *If M is a bounded-above A-module of finite type, then the epimorphism $\Sigma P \to Z_2$ induces isomorphisms*

$$\mathrm{Tor}_{s,t}^A(Z_2, \Sigma P \otimes M) \to \mathrm{Tor}_{s,t}^A(Z_2, M),$$

$$\mathrm{Ext}_A^{s,t}(M, Z_2) \to \mathrm{Ext}_A^{s,t}(\Sigma P \otimes M, Z_2).$$

It was observed in [6] that [5, 1.1] implies Proposition 6, proving it first for finite modules by the 5-lemma, and then observing that M in Proposition 6 is a direct limit of finite modules.

PROPOSITION 7. *Statement* 1 *implies* (i) $\mathrm{Ext}_A^{s,t}(Z_2, P_1 \otimes P_1) = 0$ *if* $t - s < 0$, *or if* $t = s + 1 > 4$, *or if* $s = t = 0$; (ii) $\mathrm{Ext}_A^{1,1}(Z_2, P_1 \otimes P_1) = Z_2$; (iii) *for* $s \geq 2$, $\mathrm{Ext}_A^{s,s}(Z_2, P_1 \otimes P_1) \approx Z_2 \oplus Z_2$ *with basis* $h_0^{s-1}a$ *and* $h_0^{s-2}b$.

PROOF. We use Corollary 5 to transform into $\mathrm{Ext}(\ , Z_2)$ and we denote $\mathrm{Ext}_A(M, Z_2)$ by $\mathrm{Ext}(M)$. We use the exact sequence

$$\to \mathrm{Ext}^{s-1,t}(P_{-1}) \to \mathrm{Ext}^{s,t}(P^{-2}) \to \mathrm{Ext}^{s,t}(P) \to \mathrm{Ext}^{s,t}(P_{-1}) \to$$

and Proposition 6 to find

$$\mathrm{Ext}^{s,t}(P^{-2} \otimes P) = \begin{cases} Z_2, & t - s = -2, s \geq 1, \\ Z_2, & t - s = -1, s = 1, 2, \\ 0, & t - s < -2, \text{or } s = 0, \text{or } t - s = -1, s \geq 3, \end{cases}$$

with the Z_2's in $t - s = -2$ related by h_0 and those in $t - s = -1$ not related by h_0. Next we use the exact Ext sequence of $P \otimes P_{-1} \to P \otimes P \to P \otimes P^{-2}$ together with Statement 1 and the above calculation of $\mathrm{Ext}(P^{-2} \otimes P)$ to find that in $t - s \leq 0$,

$$\mathrm{Ext}^{s,t}(P \otimes P_{-1}) \approx \mathrm{Ext}^{s,t}(\Sigma^{-1}(P_{-1} \oplus P_0)).$$

To show $\mathrm{Ext}^{s,t}(P \otimes P^{-2}) \to \mathrm{Ext}^{s,t}(P \otimes P)$ is nontrivial in this exact sequence when $s = 1$ and $t = -1$ and 0, we use the fact that $\{x^j \otimes x^{-1}: j \in Z\}$ generate $P \otimes P_{-1}$ over A so that $\dim(\mathrm{Ext}^{0,t}(P \otimes P_{-1})) \leq 1$.

An easy minimal resolution calculation shows that in $t - s \leq 1$, $\mathrm{Ext}^{s,t}(P_{-1} \otimes P_{-1})$ is zero except for Z_2 in $(s, t) = (0, -2)$, $(0, -1)$, $(0, 0)$, and $(1, 2)$. Since $\{x^{-1} \otimes x^j: j \leq -2\}$ generate the A-module $P_{-1} \otimes P^{-2}$, $\mathrm{Ext}^{0,t}(P_{-1} \otimes P^{-2}) = 0$ if

$t \geqslant -2$. These observations and the previous paragraph enable us to calculate the exact Ext sequence of

$$P_{-1} \otimes P_{-1} \to P_{-1} \otimes P \to P_{-1} \otimes P^{-2},$$

to obtain

$$\mathrm{Ext}^{s,t}(P_{-1} \otimes P^{-2}) = \begin{cases} Z_2, & t-s=-1, s \geqslant 1, \\ 0, & t-s<-1, \\ 0, & t-s=0, s>2. \end{cases}$$

Finally the Ext sequence of $P_{-1} \otimes P^{-2} \to P \otimes P^{-2} \to P^{-2} \otimes P^{-2}$ is easily calculated in this range, yielding the proposition. □

Let λ denote the usual generator of $[RP_1, S^0]$. [4].

COROLLARY 8. *Statement* 1 *implies* $[RP_1^\infty \wedge RP_1^\infty, S^0] \approx Z_2^\wedge \oplus Z_2^\wedge$ *with one summand generated by* $\lambda \wedge \lambda$ *and the other by a filtration* 1 *map*.

PROOF. We filter $RP_1^\infty \wedge RP_1^\infty$ by subspaces $X_l = RP_1^{2l} \wedge RP_1^{2l}$ so that $[X_l, S^0]$ is finite and, hence, $\lim^1$-terms vanish, and $[RP_1^\infty \wedge RP_1^\infty, S^0]$ is a $Z_2^\wedge$-module as in [4, p. 456]. Using an analogue of [4, 2.4], one defines a $Z_2^\wedge$-morphism $Z_2^\wedge \oplus Z_2^\wedge \to [RP_1^\infty \wedge RP_1^\infty, S^0]$, which is a homeomorphism by the argument of [4, p. 456]. □

PROPOSITION 9. $A(Z_2 \times Z_2)^\wedge \approx Z \oplus Z_2^\wedge \oplus Z_2^\wedge \oplus Z_2^\wedge \oplus Z_2^\wedge$.

PROOF. As in [3, pp. 40–47] we find $A(Z_2 \times Z_2)$ additively isomorphic to a free abelian group on five generators $1, a_1, a_2, a_3, b$, with multiplication $a_i^2 = 2a_i$, $a_i a_j = b$ if $i \neq j$, and augmentation $\varepsilon(1) = 1$, $\varepsilon(a_i) = 2$, $\varepsilon(b) = 4$. If $I = \ker \varepsilon$, then $I = \langle \{2 - a_i\} \rangle$, and for $n \geqslant 2$, $I^n = \langle \{2^{n-1}(2 - a_i)\}, 2^{n-2}(4 - b) \rangle$. There is an isomorphism

$$A(Z_2 \times Z_2)/I^n \xrightarrow{\phi} Z \oplus Z_{2^{n-1}} \oplus Z_{2^{n-1}} \oplus Z_{2^{n-1}} \oplus Z_{2^{n-2}}$$

defined by

$$\phi(c_0 + c_1 a_1 + c_2 a_2 + c_3 a_3 + c_4 b)$$
$$= (c_0 + 2c_1 + 2c_2 + 2c_3 + 4c_4, c_1 + c_2, c_2 + c_3, c_3 + 2c_4, c_4). \quad \square$$

PROOF OF THEOREM 2. Let $\tilde{A}(G)^\wedge$ denote $\ker(A(G)^\wedge \to A(1))$. Lin's theorem [4, 1.1] says that $\tilde{A}(Z_2)^\wedge \to [RP_1^\infty, S^0]$ is an isomorphism of groups isomorphic to $Z_2^\wedge$. There is a commutative diagram with rows split exact, where α' is defined by restricting $\hat{\alpha}$ to $\ker(p)$.

$$\begin{array}{ccccccccc} 0 & \to & Z_2^\wedge \oplus Z_2^\wedge & \xrightarrow{i} & A(Z_2 \times Z_2)^\wedge & \xrightarrow{p} & \tilde{A}(Z_2)^\wedge \oplus \tilde{A}(Z_2)^\wedge \oplus A(1) & \to & 0 \\ & & \downarrow \alpha' & & \downarrow \hat{\alpha} & & \downarrow \approx & & \\ 0 & \to & [RP_1^\infty \wedge RP_1^\infty, S^0] & \xrightarrow{i'} & [RP_1^\infty \times RP_1^\infty, S^0] & \xrightarrow{p'} & [RP_1^\infty, S^0] \oplus [RP_1^\infty, S^0] \oplus [S^0, S^0] & \to & 0 \end{array}$$

$\hat{\alpha}_G$ may be defined by representing a G-set S as a homomorphism $G \xrightarrow{\phi_s} \mathfrak{S}_n$ for some integer n and defining $\hat{\alpha}_G(S)$ to be the class of the composite

$$BG \xrightarrow{B\phi_s} B\mathfrak{S}_n \to \coprod_j B\mathfrak{S}_j \xrightarrow{q} \Omega_0^\infty S^\infty$$

where q is as in [8]. The generators $\tilde{m}$ and $\tilde{w}$ of $Z_2^{\wedge}\oplus Z_2^{\wedge}$ satisfy $i(\tilde{m}) = [m] - [mi_1] - [mi_2] + 2$ and $i(\tilde{w}) = [w] - [wi_1] - [wi_2] + 4$, where m: $Z_2 \times Z_2 \to \mathfrak{S}_2$ sends both T_1 and T_2 nontrivially, w: $Z_2 \times Z_2 \to \mathfrak{S}_4$ satisfies $w(T_1) = (1\ 2)(3\ 4)$ and $w(T_2) = (1\ 3)(2\ 4)$, and for $j = 1$ and 2, i_j: $Z_2 \times Z_2 \to Z_2 \times Z_2$ is defined by $i_j(T_j) = T_j$, $i_j(T_{3-j}) = 0$.

Since $\hat{\alpha}(w) = \lambda \times \lambda$ by multiplicativity, $\alpha'(w) = \lambda \wedge \lambda$. Since α' is $Z_2^{\wedge}$-linear, to show it is an isomorphism it will suffice to show $\alpha'(\tilde{m})$ has filtration 1. It is well known that if f is the composite

$$\Sigma^N RP^{N-1} \xrightarrow{\Sigma^N\lambda'} \Sigma^N \Omega^N S^N \xrightarrow{e} S^N$$

then all functional operations $\mathrm{Sq}^j_f\, g_N$ are nonzero. Since

$$(Bm)^*\colon H^*RP^\infty \to H^*(RP^\infty \times RP^\infty)$$

sends x^i to $(x_1 + x_2)^i$, $\mathrm{Sq}^j_{f\circ Bm}(g_N)$ is nonzero in $H^*(RP^\infty \wedge RP^\infty)$ unless $j - 1$ is a 2-power. $\square$

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