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MINIMAL PERIODIC ORBITS OF CONTINUOUS MAPPINGS OF THE CIRCLE

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ABSTRACT. Let f be a continuous map of the circle into itself and suppose that $n > 1$ is the least integer which occurs as a period of a periodic orbit of f. Then we say that a periodic orbit $\{p_1, \ldots, p_n\}$ is minimal if its period is n. We classify the minimal periodic orbits, that is, we describe how the map f must act on the minimal periodic orbits. We show that there are $\varphi(n)$ types of minimal periodic orbits of period n, where φ is the Euler phi-function.

1. Introduction and statement of results. Let $C(X, X)$ denote the set of continuous maps of a space X into itself. A point $p \in X$ is a *periodic point* of a map $f \in C(X, X)$ if $f^n(p) = p$ for some positive integer n. The *period* of p is the least such integer n, and the *orbit* of p is the set $\{f^k(p): k = 1, \ldots, n\}$. We refer to such an orbit as a *periodic orbit of period n*.

Let $P(f)$ denote the set of positive integers n such that f has a periodic point of period n. The following theorem for periodic orbits of maps of the closed interval I is proved in [5] (see also [3]).

THEOREM (ŠTEFAN). *Let $f \in C(I, I)$. Suppose $n \in P(f)$ where n is odd and $n > 1$, but $j \notin P(f)$ for all $j \in \{3, 5, \ldots, n-2\}$. Let $\{p_1, \ldots, p_n\}$ be a periodic orbit of f of period n with $p_1 < p_2 < \cdots < p_n$. Let $t = (n+1)/2$. Then either* (A) *or* (B) *holds*:

$$\text{(A)} \quad \begin{aligned} &f(p_{t-k}) = p_{t+k} && \text{for } k = 1, \ldots, t-1, \\ &f(p_{t+k}) = p_{t-k-1} && \text{for } k = 0, \ldots, t-2, \quad \text{and} \\ &f(p_n) = p_t. \end{aligned}$$

$$\text{(B)} \quad \begin{aligned} &f(p_{t-k}) = p_{t+k+1} && \text{for } k = 0, \ldots, t-2, \\ &f(p_{t+k}) = p_{t-k} && \text{for } k = 1, \ldots, t-1, \quad \text{and} \\ &f(p_1) = p_t. \end{aligned}$$

In this paper we obtain a similar result for periodic orbits of maps of the circle S^1. For distinct points $a, b \in S^1$, let (a, b) and $[a, b]$ denote the open and closed intervals, respectively, from a counterclockwise to b.

THEOREM A. *Let $f \in C(S^1, S^1)$. Suppose $n \in P(f)$ where $n > 1$, and $j \notin P(f)$ for all $j \in \{1, 2, \ldots, n-1\}$. Let $P = \{p_1, \ldots, p_n\}$ be a periodic orbit of f of period n*

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where $P \cap (p_k, p_{k+1}) = \varnothing$ *for* $k = 1, \ldots, n-1$ *and* $P \cap (p_n, p_1) = \varnothing$. *Then* $f(p_k) = p_{\sigma(k)}$, *where* $\sigma = \tau^t$ *with* $1 \leq t < n$, t *relatively prime to* n, *and* τ *is the permutation of* $\{1, 2, \ldots, n\}$ *given by* $\tau(k) = k + 1$ *for* $k = 1, \ldots, n-1$ *and* $\tau(n) = 1$.

This theorem will be proved in §3.

We remark that, in Theorem A, n is the smallest element of $P(f)$. Then we say that a periodic orbit P is minimal if its period is n. Therefore, Theorem A describes how a mapping $f \in C(S^1, S^1)$ must act on a minimal periodic orbit. Furthermore, by Theorem A, there are $\varphi(n)$ types of minimal periodic orbits of period n, where φ is the Euler phi-function.

Let R be the real line, C the complex numbers, and take $S^1 = \{z \in C: |z| = 1\}$. We use the universal covering $E \in C(R, S^1)$ given by $E(x) = e^{2\pi i x}$. Let $f \in C(S^1, S^1)$, and let $F \in C(R, R)$ be a lifting of f to the covering space. If F and F' are liftings of the same map f, then $F = F' + k$ for some integer k. There exists an integer N (the degree of f) such that $F(x + 1) = F(x) + N$ for all x.

The following lemma is well known. For a proof see [**4**, p. 107].

LEMMA 1. *Let* $f \in C(S^1, S^1)$ *and let* N *be the degree of* f. *Then* f *has at least* $|1 - N|$ *fixed points.*

From this lemma, if $f \in C(S^1, S^1)$ has minimal periodic orbits of period n with $n > 1$, then the degree of f is 1.

Let $f \in C(S^1, S^1)$ and suppose the degree of f is 1. Fix a lifting F of f. If p is a periodic point of f of period n and $E(x) = p$, then $F^n(x) = x + k$ for some integer k. We shall call the number k/n the rotation number of p and denote it by $\rho_F(p)$. It is easy to see that $\rho_F(p)$ does not depend on the choice of x, and that if $F' = F + m$ then $\rho_{F'}(p) = \rho_F(p) + m$. For more details on the rotation number see [**2**].

From Theorem A, it is immediate to prove the following.

COROLLARY B. *In the hypotheses of Theorem* A, *let* p_i *be a periodic point of the minimal periodic orbit* P, *and let* F *be a lifting of* f *such that* $F(x) \in [0, 1)$ *where* $E(x) = p_i$. *Then we have* $\rho_F(p_i) = t/n$.

Note that each minimal periodic orbit is realizable for a suitable rotation map of the circle.

2. Preliminary results. Let I and J be proper closed intervals on S^1 and let $f \in C(S^1, S^1)$. We say I f-covers J if, for some closed interval $K \subset I$, $f(K) = J$.

We state the following three lemmas of Block, which will be used in the next section.

LEMMA 2 (LEMMA 1 OF [**1**]). *Let* $I = [a, b]$ *be a proper closed interval on* S^1 *and let* $f \in C(S^1, S^1)$. *Suppose* $f(a) = c$ *and* $f(b) = d$ *and* $c \neq d$. *Then either* I *f-covers* $[c, d]$ *or* I *f-covers* $[d, c]$.

LEMMA 3 (LEMMA 2 OF [**1**]). *Let* $f \in C(S^1, S^1)$. *Let* I *and* J *be proper closed intervals on* S^1 *such that* I *f-covers* J. *Suppose* L *is a closed interval with* $L \subset J$. *Then* I *f-covers* L.

LEMMA 4 (LEMMA 3 OF [1]). *Let $f \in C(S^1, S^1)$. Suppose N is a proper closed interval on S^1 such that N f-covers N. Then f has a fixed point in N.*

3. Proof of Theorem A. Let $f \in C(S^1, S^1)$ and suppose that $n > 1$ is the smallest element of $P(f)$. Let $P = \{p_1, \ldots, p_n\}$ be a periodic orbit of f of period n where $P \cap (p_k, p_{k+1}) = \varnothing$ for $k = 1, \ldots, n-1$ and $P \cap (p_n, p_1) = \varnothing$. Finally, we let $I_k = [p_k, p_{k+1}]$ for $k = 1, \ldots, n-1$ and $I_n = [p_n, p_1]$.

For $n = 2$ or $n = 3$, Theorem A is immediate. Then we may assume that $n \geqslant 4$.

By Lemma 4, I_k does not f-cover I_k for all $k = 1, \ldots, n$. From Lemmas 2 and 3, I_k f-covers I_j for some $j \neq k$ and for all $k = 1, \ldots, n$. Hence for some set of distinct I_k's, $\{I_{k_1}, \ldots, I_{k_m}\}$, I_{k_i} f-covers $I_{k_{i+1}}$ for $i = 1, \ldots, m-1$ and I_{k_m} f-covers I_{k_1}, where $2 \leqslant m \leqslant n$.

Since I_{k_m} f-covers I_{k_1}, there is a closed interval $J_m \subset I_{k_m}$ such that $f(J_m) = I_{k_1}$. Similarly, there are closed intervals $J_1, \ldots, J_{m-1}$ such that, for $i = 1, \ldots, m-1$, $J_i \subset I_{k_i}$ and $f(J_i) = J_{i+1}$. It follows that $f^m(J_1) = I_{k_1}$. By Lemma 4, f^m has a fixed point in I_{k_1}. Then $m = n$. This implies that each I_k f-covers only one I_j, for some $j \neq k$. Therefore, $P \cap f(I_k) = \{f(p_k), f(p_{k+1})\}$ for $k = 1, \ldots, n-1$ and $P \cap f(I_n) = \{f(p_1), f(p_n)\}$.

In particular, we have that I_1 f-covers I_j for some $j \neq 1$. Suppose the following is true:

$$I_j = [f(p_2), f(p_1)]. \tag{1}$$

Therefore $j \neq n$. Since each I_k f-covers only one I_i, for some $i \neq k$, by continuity we have that $f(p_{j/2+1}) \in \{p_{j/2+1}, p_{j/2+3}\}$ if j is even (see Figure 1). But this is a contradiction, because $f(p_{j/2+1}) \neq p_{j/2+1}$ and $f(p_{j/2-1}) = p_{j/2+3}$. If j is odd, then we have that $f(p_{(j+3)/2}) \in \{p_{(j+1)/2}, p_{(j+5)/2}\}$ (see Figure 2). Again, this is a contradiction, since $f(p_{(j+1)/2}) = p_{(j+3)/2}$ and $f(p_{(j-1)/2}) = p_{(j+5)/2}$.

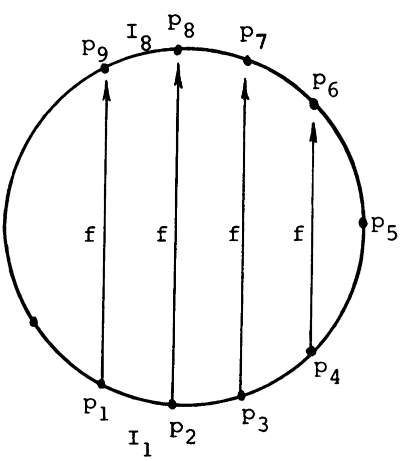

FIGURE 1

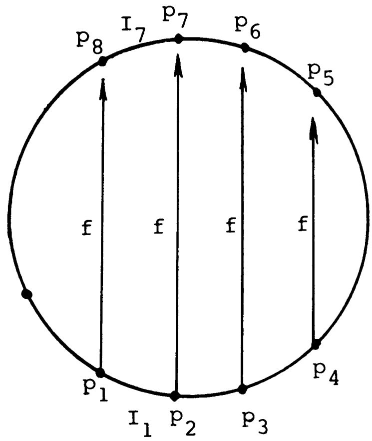

FIGURE 2

Thus the following must be true:

$$I_j = [f(p_1), f(p_2)]. \tag{2}$$

Then $f(p_1) = p_j$, $f(p_2) = p_{j+1}$ if $j \neq n$ or $f(p_2) = p_1$ if $j = n$. Since each I_k f-covers only one I_i, for some $i \neq k$, by continuity we have that there exists $t \in \{1, 2, \ldots, n-1\}$ such that $f(p_k) = p_{\sigma(k)}$ where σ is a permutation of $\{1, 2, \ldots, n\}$ such that:

(a) $\sigma(k) = k + t$ if $k + t \leqslant n$, and $\sigma(k) = k + t - n$ if $k + t > n$.

(b) σ^i is not the identity for all $i \in \{1, 2, \ldots, n-1\}$. Then Theorem A follows.

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