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STIEFEL-WHITNEY CLASSES IN $H^*BO\langle\phi(r)\rangle$

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ABSTRACT. We determine the Stiefel-Whitney classes in $H^*(BO; \mathbf{Z}_2)$ which are mapped nontrivally by the homomorphism induced by the covering projection p: $BO\langle\phi(r)\rangle \to BO$.

Let $\phi(r)$ be the function defined by $\phi(r) = 8a + 2^b$ where $r = 4a + b$ with $0 \leqslant b \leqslant 3$ and let $BO\langle\phi(r)\rangle$ be the $(\phi(r) - 1)$-connected covering of BO. It follows from Stong's computation of $H^*(BO\langle\phi(r)\rangle; \mathbf{Z}_2)$ in [4] that the covering map

$$p\colon BO\langle\phi(r)\rangle \to BO$$

maps the Stiefel-Whitney classes $w_i \in H^*(BO; \mathbf{Z}_2)$ to generators in $H^*BO\langle\phi(r)\rangle$ if $i - 1$ has at least r ones in its dyadic expansion. The remaining classes are mapped to decomposables. In this note we determine which Stiefel-Whitney classes are mapped to nonzero decomposables. In doing so we display a relationship between $H^*BO\langle\phi(r)\rangle$ and the cohomology of certain spaces related to RP^∞.

Let A denote the mod 2 Steenrod algebra and A_r the subalgebra generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \ldots, \mathrm{Sq}^{2^r}$. Let $P = \mathbf{Z}_2[x, x^{-1}]$ be the ring of Laurent polynomials in one variable x of degree $+1$, made into a module over A by setting

$$\mathrm{Sq}^i x^j = \frac{j(j-1)\cdots(j-i+1)}{1\cdot 2\cdots i}\cdot x^{i+j}.$$

Let $F_{-2,r}$ denote the A_r-submodule of P generated by x^j with $j < -2$.

THEOREM A. *The class p^*w_n is nonzero in $H^*BO\langle\phi(r)\rangle$ if and only if $\Sigma P/F_{-2,r}$ is nonzero in dimension n. The Poincaré series for $\Sigma P/F_{-2,r}$ is*

$$\frac{1}{1-t^{2^{r+1}}}(1+t^{2^r})(1+t^{3\cdot 2^{r-1}})\cdots(1+t^{(2^r-1)\cdot 2})(1+t^{2^{r+1}-1}).$$

The theorem is a consequence of the following two lemmas and the fact (from [2]) that as $\mathbf{Z}_2$-vector spaces

$$\Sigma P/F_{-2,r} \cong \bigoplus_{j\equiv 0\,(2^{r+1})} \Sigma^j(A_r \otimes_{A_{r-1}} \mathbf{Z}_2).$$

LEMMA 1. *The class p^*w_n is nonzero in $H^*BO\langle\phi(r)\rangle$ if and only if $A \otimes_{A_{r-1}} \mathbf{Z}_2$ is nonzero in dimension n.*

LEMMA 2. *$A \otimes_{A_{r-1}} \mathbf{Z}_2$ and $\bigoplus_{j\equiv 0\,(2^{r+1})} \Sigma^j(A_r \otimes_{A_{r-1}} \mathbf{Z}_2)$ are nonzero in exactly the same dimensions.*

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PROOF OF LEMMA 1. Lemma 1 follows as a corollary to the following theorem which we prove by using a slight generalization of an argument of Giambalvo [1].

THEOREM B. *The map* $e\colon A \otimes_{A_{r-1}} \mathbf{Z}_2 \to H^*MO\langle\phi(r)\rangle$ *given by evaluation on the Thom class* $U \in H^*MO\langle\phi(r)\rangle$ *is a monomorphism.*

As remarked in [1] it suffices to prove that e is a monomorphism on the primitive elements of $A \otimes_{A_{r-1}} \mathbf{Z}_2$. Since $\chi(A \otimes_{A_{r-1}} \mathbf{Z}_2)^* \cong \mathbf{Z}_2[\xi_1^{2^r}, \xi_2^{2^{r-1}}, \ldots, \xi_{r+1}, \ldots]$, where ξ_i is the Milnor basis element of A^* in dimension $2^i - 1$; there are primitives in $A \otimes_{A_{r-1}} \mathbf{Z}_2$ only in degrees 2^r, $3 \cdot 2^{r-1}$, $7 \cdot 2^{r-2}, \ldots, 2^{r+1} - 2$ and $2^i - 1$ for $i \geqslant r + 1$. For purely dimensional reasons the first $r + 1$ primitives must be Sq^{2^r}, $\mathrm{Sq}^{3 \cdot 2^{r-1}}, \ldots, \mathrm{Sq}^{2^{r+1}-2}$. The remaining primitives Q^{2^i-1}, $i \geqslant r + 1$, are projections of primitives in A. Now

$$\mathrm{Sq}^j U = w_j \cdot U \quad \text{for } j = 2^r, 3 \cdot 2^{r-1}, \ldots, 2^{r+1} - 2$$

and

$$Q^{2^i-1} U = w_{2^i-1} \cdot U + (\text{decomposables}) \cdot U \quad \text{for } i \geqslant r + 1.$$

Since the numbers $j - 1$ and $2^i - 2$ for $i \geqslant r + 1$ all have at least r ones in their dyadic expansion, $\mathrm{Sq}^j U$ and $Q^{2^i-1}U$ are nonzero by Stong's result, proving that e is a monomorphism. To deduce the lemma we need to show that Sq^n is nonzero in $A \otimes_{A_{r-1}} \mathbf{Z}_2$ if and only if there is a monomial of dimension n in $\chi(A \otimes_{A_{r-1}} \mathbf{Z}_2)^*$. To see this recall that there is a right A-module structure on A^*, given by the duality pairing, with the property that

$$\xi_k \chi \mathrm{Sq}^j = \begin{cases} \xi_s & \text{if } j = 2^k - 2^s, \\ 0 & \text{otherwise.} \end{cases}$$

Since χ commutes with the diagonal homomorphism in A, we have

$$\langle (\xi_1^{\varepsilon_1} \xi_2^{\varepsilon_2} \cdots \xi_t^{\varepsilon_t}), \chi \mathrm{Sq}^n \rangle = 1 \quad \text{if } n = \varepsilon_1 + 3\varepsilon_2 + \cdots + (2^t - 1)\varepsilon_t.$$

Considering the induced right $\chi(A \otimes_{A_{r-1}} \mathbf{Z}_2)$-module structure on $\chi(A \otimes_{A_{r-1}} \mathbf{Z}_2)^*$ then yields the result that $\chi \mathrm{Sq}^n$ is nonzero in $\chi(A \otimes_{A_{r-1}} \mathbf{Z}_2)$ if and only if there is a monomial of dimension n in $\chi(A \otimes_{A_{r-1}} \mathbf{Z}_2)^*$ completing the proof of Lemma 1.

PROOF OF LEMMA 2. By only a slight modification of the argument given by Peterson, in [3], to compute $\chi(A \otimes_{A_1} \mathbf{Z}_2)^*$ and the fact that

$$A_r^* = A^* / \left(\xi_1^{2^{r+1}}, \xi_2^{2^r}, \ldots, \xi_{r+1}^2, \xi_{r+2}, \ldots\right),$$

we can show that as $\mathbf{Z}_2$-vector spaces

$$\chi(A_r \otimes_{A_{r-1}} \mathbf{Z}_2)^* \cong \Lambda\left(\xi_1^{2^r}, \xi_2^{2^{r-1}}, \ldots, \xi_r^2, \xi_{r+1}\right)$$

where the right-hand side is the exterior algebra over $\mathbf{Z}_2$ generated by $\xi_1^{2^r}, \ldots, \xi_{r+1}$. We define a map $\lambda\colon \Sigma^{2^{r+1}t}\chi(A_r \otimes_{A_{r-1}} \mathbf{Z}_2) \to \chi(A \otimes_{A_{r-1}} \mathbf{Z}_2)^*$ of vector spaces over $\mathbf{Z}_2$, by

$$\lambda\left(\xi_1^{2^r\varepsilon_1} \xi_2^{2^{r-1}\varepsilon_2} \cdots \xi_{r+1}^{\varepsilon_{r+1}}\right) = \xi_1^{2^r(\varepsilon_1 + 2t)} \xi_2^{2^{r-1}\varepsilon_2} \cdots \xi_{r+1}^{\varepsilon_{r+1}}$$

for $\varepsilon_i = 0$ or 1. Since λ is a monomorphism it follows that $A \otimes_{A_{r-1}} \mathbf{Z}_2$ is nonzero in dimension n if $\bigoplus_{j\equiv 0\,(2^{r+1})} \Sigma^j A_r \otimes_{A_{r-1}} \mathbf{Z}_2$ is. Conversely we define a map

$$\rho\colon \chi(A \otimes_{A_{r-1}} \mathbf{Z}_2)^* \to \bigoplus_{j\equiv 0\,(2^{r+1})} \Sigma^j \chi(A_r \otimes_{A_{r-1}} \mathbf{Z}_2)^*$$

by the following procedure. Suppose that

$$\xi_1^{2^r a_1} \xi_2^{2^{r-1} a_2} \cdots \xi_{r+1}^{a_{r+1}} \cdots \xi_j^{a_j}, \qquad j > r+1,\ a_i \in \mathbf{Z},$$

is a monomial in $\chi(A \otimes_{A_{r-1}} \mathbf{Z}_2)^*$ of dimension n. Begin by replacing it with the monomial

$$\xi_1^{2^r a_1 + 2^{r+1} v} \xi_2^{2^{r-1} a_2} \cdots \xi_{r+1}^{\omega} \tag{3}$$

where $\omega = \sum_{i=r+1}^{j} a_i$ and $v = a_{r+2} + 3a_{r+3} + 7a_{r+4} + \cdots + (2^{j-r-1} - 1)a_j$. This monomial is also of dimension n. Next we inductively "reduce" the monomial while at the same time preserving its dimension. If (3) is of the form

$$\xi_1^{2^r b_1} \xi_2^{2^{r-1} b_2} \cdots \xi_i^{2^{r+1-i} b_i} \xi_{i+1}^{2^{r-i} \varepsilon_{i+1}} \cdots \xi_{r+1}^{\varepsilon_{r+1}}$$

with $i \leqslant r+1$, $b_i \in \mathbf{Z}$ and each ε_j either 0 or 1 we replace it with the monomial

$$\begin{aligned} &\xi_1^{2^r(b_1+t) + 2^{r+1} c} \xi_2^{2^{r-1}(b_2+\varepsilon_2)} \xi_3^{2^{r-2}(b_3+\varepsilon_3)} \cdots \\ &\qquad\qquad \xi_{i-1}^{2^{r+2-i}(b_{i-1}+\varepsilon_{i-1})} \xi_i^{2^{r+1-i}\varepsilon_i} \xi_{i+1}^{2^{r-i}\varepsilon_{i+1}} \cdots \xi_{r+1}^{\varepsilon_{r+1}} \end{aligned} \tag{4}$$

where $b_i = \varepsilon_i + 2\varepsilon_{i-1} + 4\varepsilon_{i-2} + \cdots + 2^{i-2}\varepsilon_2 + 2^{i-1}t$, each ε_j is either 0 or 1 and $c = \varepsilon_{i-1} + 3\varepsilon_{i-2} + \cdots + (2^{i-2} - 1)\varepsilon_2 + (2^{i-1} - 1)t$. The monomial (4) also has dimension n. Continuing in this fashion we end up with a monomial of the form

$$\xi_1^{2^r \varepsilon_1 + 2^{r+1} q} \xi_2^{2^{r-1} \varepsilon_2} \xi_3^{2^{r-2} \varepsilon_3} \cdots \xi_{r+1}^{\varepsilon_{r+1}} \tag{5}$$

with each ε_j zero or one. We define $\rho(\xi_1^{2^r a_1} \xi_2^{2^{r-1} a_2} \cdots \xi_j^{a_j})$ to be the nonzero monomial $\xi_1^{2^r \varepsilon_1} \cdots \xi_{r+1}^{\varepsilon_{r+1}}$ in $\Sigma^{2^{r+1} q} \chi(A_r \otimes_{A_{r-1}} \mathbf{Z}_2)^*$ of dimension n; so

$$\bigoplus_{j\equiv 0\,(2^{r+1})} \Sigma^j \chi(A_r \otimes_{A_{r-1}} \mathbf{Z}_2)$$

is nonzero in dimension n completing the proof of Lemma 2.

ADDED IN PROOF. Related results about the vanishing of Stiefel-Whitney classes have been proved by R. Stong. See §3 of *Cobordism and Stiefel-Whitney Numbers*, Topology **4** (1965), 241–246.

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