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SHORTER NOTES

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A SHORT PROOF OF THE JUNNILA QUASI-METRIZATION THEOREM

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ABSTRACT. Junnila has shown in [2] that the classic γ-space conjecture is true in the class of developable spaces. This paper presents a new and straightforward proof of Junnila's theorem, that every developable γ-space is quasi-metrizable.

All spaces are T_1 topological spaces. Following [1], a *neighbournet* V on a space X is a binary relation on X such that, for each $x \in X$, the set $V\{x\} = \{y \in X \mid (x,y) \in V\}$ is a neighbourhood of x. Then X is a γ-space iff there exists a decreasing sequence $\{V_n \mid n \in \mathbf{N}\}$ of neighbournets (which will be called a γ-*sequence*) such that, at each point $x \in X$, the family $\{V_n^2\{x\} \mid n \in \mathbf{N}\}$ is a neighbourhood base [3] and [1]. Similarly, X is a quasi-metrizable space iff there exists a sequence $\{W_n \mid n \in \mathbf{N}\}$ of neighbournets (called a *normal basic sequence*) such that $W_{n+1}^2 \subseteq W_n$ for each $n \in \mathbf{N}$ and, at each point $x \in X$, the family $\{W_n\{x\} \mid n \in \mathbf{N}\}$ is a neighbourhood base [1].

LEMMA. *Let U be a neighbournet on a developable γ-space X. Then there exists another neighbournet W on X such that $W^4 \subseteq U^2$.*

PROOF. Let $\{\mathcal{D}_n \mid n \in \mathbf{N}\}$ be a development for X, and $\{V_n \mid n \in \mathbf{N}\}$ a γ-sequence. Then $\{V_n^5\{x\} \mid n \in \mathbf{N}\}$ will be a neighbourhood base at x. Without loss of generality, assume $\mathcal{D}_{n+1}$ refines $\mathcal{D}_n$.

For each $x \in X$ define

$$m(x) = \min\{n \in \mathbf{N} \mid V_n^5\{x\} \subseteq U\{x\}\};$$

$$k(x) = \min\{n \geqslant m(x) \mid \mathrm{St}(x, \mathcal{D}_n) \subseteq V_{m(x)}\{x\}\};$$

$$l(x) = \max\{\min\{k(y) \mid y \in A\} \mid A \text{ is a neighbourhood of } x\} \leqslant k(x).$$

Let $C_0 = \varnothing$ and $C_n = \{x \in X \mid k(x) \leqslant n\}$; then $\overline{C}_n = \{x \in X \mid l(x) \leqslant n\}$. Obviously, $x \notin \overline{C}_{l(x)-1}$.

Define W by $W\{x\} = V_{k(x)}\{x\} \setminus \overline{C}_{l(x)-1}$.

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Now suppose $x_4 \in W^4\{x_0\}$, and find $x_1, x_2, x_3 \in X$ so that $x_{i+1} \in W\{x_i\}$ for $i \leqslant 3$. Since $x_{i+1} \notin \overline{C}_{l(x_i)-1}$ we have $l(x_0) \leqslant l(x_i)$ for all i. There exists $y \in U\{x_0\} \cap \mathrm{St}(x_0, \mathfrak{D}_{l(x_0)})$ such that $k(y) = l(x_0)$. Therefore $m(y) < k(y) = l(x_0) < l(x_i) \leqslant k(x_i)$, and so

$$x_{i+1} \in W\{x_i\} \subseteq V_{k(x_i)}\{x_i\} \subseteq V_{m(y)}\{x_i\}.$$

Thus $x_4 \in V^4_{m(y)}\{x_0\}$, and as $y \in \mathrm{St}(x_0, \mathfrak{D}_{l(x_0)})$ then

$$x_0 \in \mathrm{St}(y, \mathfrak{D}_{l(x_0)}) = \mathrm{St}(y, \mathfrak{D}_{k(y)}) \subseteq V_{m(y)}\{y\};$$

hence $x_4 \in V^5_{m(y)}\{y\} \subseteq U\{y\}$.

But $y \in U\{x_0\}$; hence $x_4 \in U^2\{x_0\}$ as required.

THEOREM (JUNNILA [2]). *Every developable γ-space is quasi-metrizable.*

PROOF. Let X be a developable γ-space, and $\{V_n | n \in \mathbf{N}\}$ a γ-sequence. Using the above lemma, inductively define a sequence $\{W_n | n \in \mathbf{N}\}$ of neighbournets so that $W_1^4 \subseteq V_1^2$ and $W_{n+1}^4 \subseteq (W_n \cap V_{n+1})^2$. Then $\{W_n^2 | n \in \mathbf{N}\}$ is a normal basic sequence, and hence X is quasi-metrizable.

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