PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, December 1981

THE DISTRIBUTION FUNCTION IN THE MORREY SPACE

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ABSTRACT. For $1 < p < \infty$, we consider p-integrable functions on a finite cube Q_0 in $\mathbf{R}^n$, satisfying

$$(1) \qquad \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p \, dx\right)^{1/p} \leqslant C\varphi(|Q|)$$

for every parallel subcube Q of Q_0, where $|Q|$ denotes the volume of Q, f_Q is the mean value of f over Q and $\varphi(t)$ is a nonnegative function defined in $(0, \infty)$, such that $\varphi(t)$ is nonincreasing near zero, $\varphi(t) \to \infty$ as $t \to 0$, and $t\varphi^p(t)$ is nondecreasing near zero. The constant C does not depend on Q. Let g be a nonnegative p-integrable function $g\colon (0, 1) \to \mathbf{R}$ such that g is nonincreasing and $g(t) \to \infty$ as $t \to 0$. We prove here that there exist a cube Q_0 and a function f satisfying condition (1) for every parallel subcube Q of Q_0, such that $\delta_f(\lambda) \geqslant C_1\delta_g(\lambda)$ for $\lambda \geqslant \lambda_0$, $C_1 > 0$, where $\delta(\lambda)$ denotes the distribution function.

John and Nirenberg have introduced in [**1**] the functions of bounded mean oscillation. An integrable function f on a finite cube Q_0 in $\mathbf{R}^n$ is called a BMO function if there exists a constant $C > 0$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q| \, dx \leqslant C$$

for every parallel subcube Q of Q_0, where $|Q|$ denotes the volume of Q and f_Q is the mean value of f over Q.

These authors have shown that the distribution function of a function of bounded mean oscillation decreases exponentially. More exactly, there exist constants $C, \alpha > 0$ such that

$$\delta_{f-f_{Q_0}}(\lambda) = \operatorname{meas}\{x \in Q_0 \mid |f(x) - f_{Q_0}| > \lambda\} \leqslant Ce^{-\alpha\lambda}|Q_0| \quad \text{for every } \lambda > 0.$$

This implies that the BMO functions satisfy additional conditions. Actually, they belong to $L^p(Q_0)$ for all $p < \infty$ and they satisfy

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p \, dx\right)^{1/p} \leqslant C.$$

Presented at the meeting of the Unión Matemática Argentina, August 1980; received by the editors October 20, 1980.

AMS (MOS) subject classifications (1970). Primary 43A15, 46E30; Secondary 26A09.

Key words and phrases. Mean oscillation, distribution function, Morrey space.

Now, we are led to consider more general spaces, for instance, those p-integrable functions f on Q_0 satisfying

$$(2)\qquad \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p\, dx\right)^{1/p} \leqslant C|Q|^{-\alpha} \quad \text{for some } 0 < \alpha < 1/p.$$

It would be interesting to obtain some estimate for the distribution function in this space. However, it does not seem to be possible. Actually, we show this in a more general space.

In fact, let $\varphi(t)$ be a nonnegative function defined in $(0, \infty)$ such that $\varphi(t)$ does not increase near zero, $\varphi(t) \to \infty$ as $t \to 0$, and $t\varphi^p(t)$ does not decrease near zero. We say that a function f that is p-integrable over a cube Q_0 in $\mathbf{R}^n$ belongs to the space $M_\varphi^p(Q_0)$ if it satisfies $(|Q|^{-1}\int_Q |f(x) - f_Q|^p\, dx)^{1/p} \leqslant C\varphi(|Q|)$ for every parallel subcube Q of Q_0.

When $\varphi(t) = t^{-\alpha}$, $0 < \alpha < 1/p$, we get the Morrey space, that is the functions satisfying (2). Now, we prove the following result.

PROPOSITION. *Let g: $(0, 1) \to \mathbf{R}$ be a nonnegative, nonincreasing p-integrable function such that $g(t) \to \infty$ as $t \to 0$. Then, there exist a cube Q_0, a function $f \in M_\varphi^p(Q_0)$ and two constants $C_1, \lambda_0 > 0$ such that*

$$\delta_f(\lambda) \geqslant C_1\delta_g(\lambda) \quad \textit{for } \lambda \geqslant \lambda_0.$$

PROOF. First, we prove the assertion in one variable. The general case will follow from this one.

According to the hypothesis, there exists $a \leqslant 1$ such that $\varphi(t)$ does not increase for $t \leqslant a$, $t\varphi^p(t)$ does not decrease there, and $\varphi(a) > 0$. We can also suppose that $\varphi(a) = 1$, so we get $\varphi(t) \geqslant 1$ for $t \leqslant a$. We suppose also that $\int_0^1 g(t)^p\, dt \leqslant a/4^p$. On the other hand, we complete the definition of the function g in such a way that it remains left continuous.

Let $h \geqslant 2$ be the first natural number such that $g(t) < 2^{h-1}$, for some t. For each $k \geqslant h + 1$, we consider the interval $I_k = (x_k, x_{k-1})$, such that $2^{k-2} \leqslant g(t) < 2^{k-1}$ in I_k.

We assert that the sequence $\{x_k\}$ converges to zero. In fact, since it is decreasing, it has a limit $L \geqslant 0$. L must be zero because, by construction, we have $g(x_k) \geqslant 2^{k-2}$, $k \geqslant h + 1$. Furthermore, the length of the intervals I_k decreases as a geometric progression. In fact,

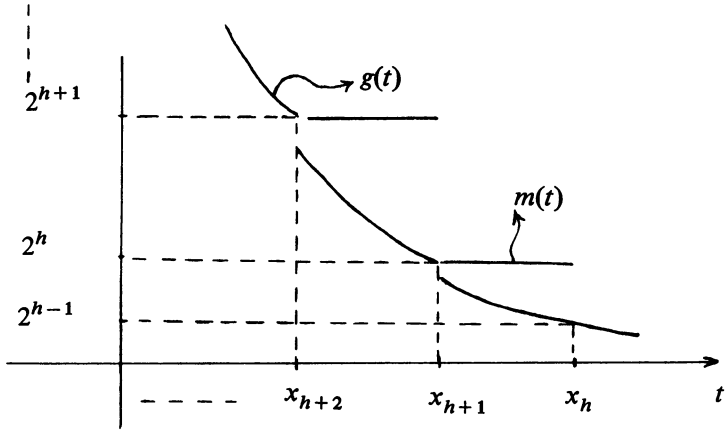
$$|I_k| 2^{p(k-2)} \leqslant \int_{I_k} g(t)^p\, dt \leqslant \frac{a}{4^p},$$

so that $|I_k| \leqslant a/2^{pk}$ for $k \geqslant h + 1$.

Now, we define a step function $m(t)$ as

$$m(t) = 2^{k-1} \quad \text{in } I_k, \text{ for } k \geqslant h + 1.$$

Clearly, $\delta_m(\lambda) \geqslant \delta_g(\lambda)$ for $\lambda \geqslant 2^{h-1}$. Now, we will replace each interval I_k by another one, $J_k = (y_k, y_{k-1})$, of length $|J_k| = 2^{pk}|I_k| = 2^p \int_{I_k} m(t)^p\, dt$.

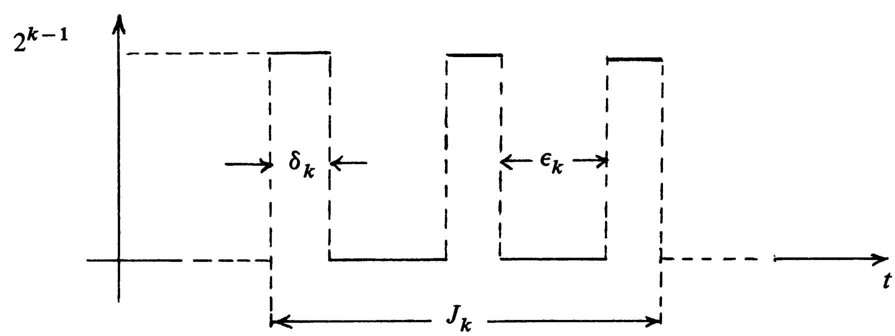


We assert that $\Sigma_{k>h+1}|J_k| < \infty$. In fact,

$$\sum_{k>h+1} |J_k| = \sum_{k>h+1} 2^{pk}|I_k| = 4^p \sum_{k>h+1} 2^{p(k-2)}|I_k|$$

$$\leqslant 4^p \sum_{k>h+1} \int_{I_k} g(t)^p\, dt \leqslant 4^p \int_0^1 g(t)^p\, dt \leqslant a.$$

Now, we will define a function $f(t)$ on the interval $(0, y_h)$ in the following way. We fix one interval I_k and we divide it into n_k subintervals of length $\delta_k = |I_k|/n_k$. The number n_k will be selected later. Now, let $\varepsilon_k = (|J_k| - |I_k|)/(n_k - 1)$. We divide the interval J_k into $2n_k - 1$ subintervals of length δ_k and ε_k alternatively. We define $f(t)$ as 2^{k-1} in the intervals of length δ_k and zero in the others.



Over each interval J_k the measure of the set where $f(t)$ does not vanish is exactly $|I_k|$. Furthermore, $f(t)$ coincides with $m(t)$ on that set. So, both functions have the same distribution function.

Now, we assert that selecting the number n_k in each interval J_k in a correct way, we get $f(t) \in M_\varphi^p((0, y_h))$. Actually, we will prove that there exists a constant $C > 0$ such that

$$\int_J f(t)^p\, dt \leqslant C|J|\varphi^p(|J|)$$

for every subinterval J of $(0, y_h)$. This will clearly imply that $f \in M_\varphi^p((0, y_h))$.

First, let us consider the interval $J = (0, y_{L-1})$, for some L,

$$\int_J f(t)^p\,dt = \sum_{k>L} \int_{J_k} f(t)^p\,dt = \sum_{k>L} 2^{p(k-1)}|I_k|$$
$$= 2^{-p} \sum_{k>L} |J_k| = 2^{-p}|J|.$$

As was shown above, $\sum_{k>L}|J_k| \leqslant a$; thus, $\varphi(|J|) \geqslant 1$ or $|J|\varphi^p(|J|) \geqslant |J|$. So, we get

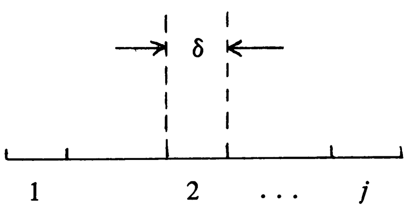
$$\int_J f(t)^p\,dt \leqslant 2^{-p}|J|\varphi^p(|J|).$$

Now, let us consider an interval $J = (y_s, y_{L-1})$. In the same way, we obtain

$$\int_J f(t)^p\,dt = \sum_L^s \int_{J_k} f(t)^p\,dt = \sum_L^s 2^{p(k-1)}|I_k|$$
$$= 2^{-p}\sum_L^s |J_k| = 2^{-p}|J| \leqslant 2^{-p}|J|\varphi^p(|J|).$$

Now, we will consider an interval J contained in one of the intervals J_k. We will select the number n_k in order to obtain the desired inequality over this interval.

Since we have supposed the index k to be fixed, we will write simply n, δ, ε. We first assume that there are j intervals of length δ, which cover the interval J, in the following sense.



It would be desirable to obtain the inequality

$$2^{p(k-1)}j\delta \leqslant (j\delta + (j-1)\varepsilon)\varphi^p(j\delta + (j-1)\varepsilon), \qquad 1 \leqslant j \leqslant n.$$

Since $\delta = |I_k|/n$, $\varepsilon = (|J_k| - |I_k|)/n - 1 = (2^{pk} - 1)|I_k|/n - 1$, we can write the above inequality in the form

$$2^{p(k-1)}j\frac{|I_k|}{n}$$
$$\leqslant \left(j\frac{|I_k|}{n} + (j-1)(2^{pk}-1)\frac{|I_k|}{n-1}\right)\varphi^p\left(j\frac{|I_k|}{n} + (j-1)(2^{pk}-1)\frac{|I_k|}{n-1}\right)$$

or,

$$2^{p(k-1)} \leqslant \left(1 + \frac{n}{j}\,\frac{j-1}{n-1}(2^{pk}-1)\right)\varphi^p\left(|I_k|\left(\frac{j}{n} + \frac{j-1}{n-1}(2^{pk}-1)\right)\right).$$

As we saw above, $|I_k| \leqslant a/2^{pk}$. Moreover, $(j-1)/(n-1) \leqslant j/n$. Thus,

$$|I_k|\left(\frac{j}{n} + \frac{j-1}{n-1}(2^{pk}-1)\right) \leqslant \frac{a}{2^{pk}}\left(\frac{j}{n} + \frac{j}{n}(2^{pk}-1)\right) = a\frac{j}{n} \leqslant a.$$

Since we have supposed that φ is a nonincreasing function for $t < a$, we obtain

$$\varphi\left(|I_k|\left(\frac{j}{n} + \frac{j-1}{n-1}(2^{pk} - 1)\right)\right) \geqslant \varphi\left(a\frac{j}{n}\right) \geqslant \varphi(a) = 1.$$

Thus, it suffices to find a natural number n so that

$$2^{p(k-1)} \leqslant \left(1 + \frac{n}{j}\,\frac{j-1}{n-1}(2^{pk} - 1)\right)\varphi^p\left(a\frac{j}{n}\right), \qquad 1 \leqslant j \leqslant n,$$

for $k \geqslant h + 1$ fixed. Since $\varphi(t) \to \infty$ as $t \to 0$, there exists $0 < r(k,p) < a$ such that $aj/n \leqslant r$ which implies $2^{p(k-1)} \leqslant \varphi^p(aj/n)$. Thus, when $j/n \leqslant r/a$, we get the desired inequality.

Now, we suppose $1 \leqslant n/j \leqslant a/r$, and we will select n in such a way that

$$2^{p(k-1)} \leqslant 1 + \frac{n}{j}\,\frac{j-1}{n-1}(2^{pk} - 1).$$

Since $(j-1)/j = 1 - 1/j$ increases as j increases, the worst case occurs when $n/j = a/r$; that is,

$$2^{p(k-1)} \leqslant 1 + \frac{a}{r}\,\frac{nr/a - 1}{n-1}(2^{pk} - 1).$$

From this inequality, we deduce that selecting $n \geqslant (a/r - \theta)/(1 - \theta)$, where $\theta = (2^{p(k-1)} - 1)/(2^{pk} - 1)$, we obtain the desired inequality for the subinterval J. Now, we suppose that J is contained in one of the intervals of length δ_k, for k fixed.

In this case, it suffices to satisfy the inequality

$$2^{p(k-1)}|J| \leqslant |J|\varphi^p(|J|),$$

or, $2^{k-1} \leqslant \varphi(|J|)$.

Since $|J| \leqslant \delta_k$, we will have $\varphi(|J|) \geqslant \varphi(\delta_k)$; so that, it suffices to obtain $2^{k-1} \leqslant \varphi(\delta_k)$. But this is the inequality above, for $j = 1$.

In the same way, we can prove the inequality for a given subinterval J of some interval J_k. We merely have to use that the function $t\varphi^p(t)$ does not decrease for $t \leqslant a$.

Finally, let us consider a subinterval J of the interval $(0, y_h)$. We can divide J into at most three intervals. One of them is a union of some intervals J_k, and the others are contained in some other intervals $J_{k'}$ and $J_{k''}$. Thus, according to all we have said above, and using again the fact that $t\varphi^p(t)$ is a nondecreasing function, we obtain the inequality. This concludes the one variable case.

In the general case, we argue as follows. Let $f(t)$ be a function in the space $M_\varphi^p((0, y_h))$, satisfying the desired hypothesis. Let $Q_0 = \{(t_1, \ldots, t_n) | 0 < t_j < y_h, j = 1, \ldots, n\}$. We define the function $F(t_1, \ldots, t_n)$ as

$$F(t_1, \ldots, t_n) = f(t_1).$$

We assert that $F \in M_\varphi^p(Q_0)$.

In fact, let Q be a parallel subcube of Q_0; we can write $Q = S_1 \times \cdots \times S_n$, where S_j are subintervals of the same length of $(0, y_h)$. Thus,

$$\int_Q F(t_1, \ldots, t_n)^p \, dt_1 \cdots dt_n = |S_2| \cdots |S_n| \int_{S_1} f(t_1)^p \, dt_1 \leqslant C|Q|\varphi^p(|S_1|).$$

Since $|S_j| \leqslant y_h \leqslant a \leqslant 1$, we get $|Q| = |S_1| \cdots |S_n| \leqslant |S_1| \leqslant a$. So that

$$\varphi(|S_1|) \leqslant \varphi(|Q|) \quad \text{or} \quad \varphi^p(|S_1|) \leqslant \varphi^p(|Q|).$$

On the other hand, we have also that $\delta_F(\lambda) = y_h^{n-1}\delta_f(\lambda)$ for $\lambda > 0$. This completes the proof.

REMARK. In [2], the definition of the Morrey space appears in a slightly different way. Working over cubes, that definition may be stated as follows.

A p-integrable function f on a finite cube Q_0 in $\mathbf{R}^n$ belongs to the Morrey space of order α, $0 < \alpha < 1/p$, if

$$\sup_{\substack{x \in \bar{Q}_0 \\ |Q(x)| \leqslant |Q_0|}} \inf_{c \in \mathbf{C}} \left[|Q(x)|^{p\alpha - 1} \int_{Q(x) \cap Q_0} |f(y) - c|^p \, dy \right]^{1/p} < \infty$$

where $\bar{Q}_0$ means the closure of Q_0 and, given $x \in \bar{Q}_0$, $Q(x)$ is a cube centered in x, parallel to Q_0.

Actually, Campanato has shown that it is the same to consider

$$\sup_{\substack{x \in \bar{Q}_0 \\ |Q(x)| \leqslant |Q_0|}} \left[|Q(x)|^{p\alpha - 1} \int_{Q(x) \cap Q_0} |f(y)|^p \, dy \right]^{1/p} < \infty$$

(see [3]). From this, we are led to consider those p-integrable functions on Q_0 such that

$$\int_{Q(x) \cap Q_0} |f(y)|^p \, dy \leqslant C|Q(x)|\varphi^p(|Q(x)|) \quad \text{for all } x \in \bar{Q}_0,\ |Q(x)| \leqslant |Q_0|.$$

The function $F(t_1, \ldots, t_n)$ constructed in the above proposition satisfies this inequality, in fact let $Q(x) \cap Q_0 = S_1 \times \cdots \times S_n$, where S_j are subintervals of $(0, y_h)$. Then

$$\int_{Q(x) \cap Q_0} F(t_1, \ldots, t_n)^p \, dt_1 \cdots dt_n = |S_2| \cdots |S_n| \int_{S_1} f(t_1)^p \, dt_1$$
$$\leqslant C|Q(x) \cap Q_0|\varphi^p(|S_1|).$$

Since $|S_j| \leqslant y_h \leqslant a \leqslant 1$, we have $a \geqslant |S_1| \geqslant |S_1| \cdots |S_n| = |Q(x) \cap Q_0|$. The function $\varphi(t)$ is nonincreasing and the function $t\varphi^p(t)$ is nondecreasing for $t \leqslant a$, so we get

$$|Q(x) \cap Q_0|\varphi^p(|S_1|) \leqslant |Q(x) \cap Q_0|\varphi^p(|Q(x) \cap Q_0|) \leqslant |Q(x)|\varphi^p(|Q(x)|).$$

Actually, we have proved that

$$\int_{Q(x) \cap Q_0} F(t_1, \ldots, t_n)^p \, dt_1 \cdots dt_n \leqslant C|Q(x) \cap Q_0|\varphi^p(|Q(x) \cap Q_0|).$$

This concludes the remark.

ACKNOWLEDGEMENT. We thank Professor A. P. Calderón for his very helpful suggestions.

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