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ON THE UNIFORM ERGODIC THEOREM OF LIN

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ABSTRACT. Lin has given necessary and sufficient conditions for convergence in the uniform operator topology of $A_n = (I + T + \cdots + T^{n-1})/n$, T being a Banach space operator satisfying $\|T^n\|/n \to 0$. We prove a generalization in which the Cesàro means are replaced by any bounded sequence in the affine hull converging uniformly to invariance. In the case where T: $C_0(X) \to C_0(X)$ is a transient Feller operator for noncompact locally compact Hausdorff space X, we show that $\{A_n\}$ converges strongly but never uniformly.

1. Introduction. The following problem was mentioned by Robert C. Sine in correspondence with the present author several years ago. Suppose a Banach space operator satisfies the conditions of the strong ergodic theorem. When is 1 isolated in the spectrum a sufficient condition for the uniform ergodic theorem? Put another way, when 1 is isolated the resolvent has either a simple pole or else an essential singularity at 1; when is the latter case excluded? We obtain here a partial result. Namely, the resolvent of a transient Feller operator on a noncompact locally compact Hausdorff state space has a singularity at 1 which is never a simple pole. The author has not been able to construct an example of such an operator where 1 is isolated in the spectrum.

2. The ergodic theorems of Sine and of Lin. With $\mathfrak{B}(\mathfrak{X})$ the bounded linear operators on a complex Banach space $\mathfrak{X}$, let $T \in \mathfrak{B}(\mathfrak{X})$ be given with spectral radius $r(T) = 1$. We abbreviate [uniform] [strong] operator topology to [UOT] [SOT]. Let $\mathfrak{P}_1$ consist of those $P \in \mathfrak{B}(\mathfrak{X})$ which can be expressed as $P = \sum_{j=0}^{\infty} a_j T^j$ where $\sum_0^\infty a_j z^j$ has a radius of convergence greater than 1 and $\sum_0^\infty a_j = 1$. It will always be assumed that T satisfies condition UI:

(UI) For some $1 \leqslant M_0 < \infty$ the UOT closure of the set $\{(I - T)P: P \in \mathfrak{P}_1$ and $\|P\| \leqslant M_0\}$ contains $0 \in \mathfrak{B}(\mathfrak{X})$.

That is, $\mathfrak{P}_1$ contains bounded sequences $\{P_n\}$ converging uniformly to invariance: $\lim_n \|P_n - TP_n\| = 0$. We call such a sequence a UI sequence.

Let $\mathfrak{L} = \{x \in \mathfrak{X}: Tx = x\}$ be the invariant vectors of T, and let $\mathfrak{L}^*$ be the invariant vectors of the adjoint $T^* \in \mathfrak{B}(\mathfrak{X}^*)$. The generalized Sine Ergodic Theorem proved in [3] asserts that when UI holds the following conditions are equivalent:

(i) $\mathfrak{L}$ separates $\mathfrak{L}^*$;

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(ii) The SOT closure of $\mathcal{P}_1$ contains a projection Q onto $\mathcal{Q}$, necessarily unique, and each UI sequence is SOT convergent to Q.

(See [**4**] for a simplified proof and additional references.) After this, "Q exists" will mean that (ii) holds. The case $Q = 0$ is included; it requires $\mathcal{Q}^* = 0$ as well as $\mathcal{Q} = 0$.

The spectrum of $T \in \mathcal{B}(\mathcal{X})$ is denoted by $\sigma(T)$ or $\sigma(T; \mathcal{X})$, the resolvent set by $\rho(T)$, the range by $\mathcal{R}(T)$. The resolvent $R_\lambda(T) = (\lambda I - T)^{-1}$, $\lambda \in \rho(T)$, can be expressed as $R_\lambda(T) = \sum_{j=0}^{\infty} \lambda^{-j-1}T^j$ when $|\lambda| > 1$; we define $P_\lambda(T) = (\lambda - 1)R_\lambda(T)$, $\lambda \in \rho(T)$, and have $P_\lambda(T) \in \mathcal{P}_1$ when $|\lambda| > 1$.

The following result is phrased to address the Sine problem of §1. Some parts of the proof are straightforward generalizations of Lin's methods of [**2**], although we use also the operator calculus and spectral considerations [**1**, Chapter VII]. The assumption that Q exists simplifies most of the arguments. The subspaces $\mathcal{Q} = \mathcal{R}(Q)$ and $\mathcal{R}(I - Q)$ are invariant under T and $I - T$, since $QT = TQ$.

THEOREM 1. *Let $T \in \mathcal{B}(\mathcal{X})$ of spectral radius* 1 *be such that UI holds and Q exists. Then the following conditions are equivalent.*

[i] $R_\lambda(T) - Q/(\lambda - 1)$ *is analytic at* $\lambda = 1$;

[ii] [iii] *For* [*some*] [*each*] *sequence* $\lambda_n \downarrow 1$, $\{P_{\lambda_n}(T)\}$ *is UI and is UOT convergent to* Q;

[iv] [v] [*Some*] [*Each*] *UI sequence* $\{P_n\}$ *is UOT convergent to* Q;

[vi] *Some* $P \in \mathcal{P}_1$ *satisfies* $\|P - Q\| < 1$;

[vii] $\mathcal{R}(I - T)$ *is closed*;

[viii] *As an operator in* $\mathcal{B}(\mathcal{R}(I - Q))$, *the restriction* $T|\mathcal{R}(I - Q)$ *has spectrum* $\sigma(T; \mathcal{R}(I - Q))$ *which does not contain* 1.

PROOF. [vii] $\Leftrightarrow$ [viii]. On $\mathcal{Q}$, $I - T$ vanishes. On $\mathcal{R}(I - Q)$, $I - T$ is 1-1, since the nullspace of $I - T$ is $\mathcal{Q}$ and $\mathcal{X}/\mathcal{Q} \approx \mathcal{R}(I - Q)$. Thus a left inverse Θ exists, defined on $\mathcal{R}(I - T)$, satisfying $\Theta(I - T) = I - Q$; the operator Θ is linear but need not be bounded. In fact, Θ is bounded iff $\mathcal{R}(I - T)$ is closed in $\mathcal{X}$ [**1**, Theorem II.2.2]. It is known that $\mathcal{R}(I - T)$ is dense in $\mathcal{R}(I - Q)$ when Q exists; see [**4**]. Thus $\mathcal{R}(I - T) = \mathcal{R}(I - Q)$ iff $\mathcal{R}(I - T)$ is closed, and in this event $\Theta \in \mathcal{B}(\mathcal{X})$ exists and is determined by $\Theta(I - T) = (I - T)\Theta = I - Q$, $\Theta Q = Q\Theta = 0$. On the other hand, the restriction $(I - T)|\mathcal{R}(I - Q)$ has a bounded inverse in the Banach space $\mathcal{R}(I - Q)$ iff $0 \notin \sigma(I - T; \mathcal{R}(I - Q))$ iff $1 \notin \sigma(T; \mathcal{R}(I - Q))$.

[viii] $\Leftrightarrow$ [i]. From $(\lambda I - T)Q = Q(\lambda I - T) = (\lambda - 1)Q$ follows $R_\lambda(T)Q = QR_\lambda(T) = Q/(\lambda - 1)$, whence

$$R_\lambda(T)(I - Q) = (I - Q)R_\lambda(T) = R_\lambda(T) - Q/(\lambda - 1), \qquad \lambda \in \rho(T).$$

The operator $R_\lambda(T)(I - Q)$ vanishes on $\mathcal{Q}$; its restriction to $\mathcal{R}(I - Q)$ can be regarded as the resolvent $R_{\lambda-1}(T - I; \mathcal{R}(I - Q))$, since $(T - I)(I - Q) = (I - Q)(T - I) = T - I$ in $\mathcal{B}(\mathcal{X})$. This resolvent is analytic at $\lambda - 1 = 0$ iff $0 \notin \sigma(T - I; \mathcal{R}(I - Q))$ iff $1 \notin \sigma(T; \mathcal{R}(I - Q))$.

[i] $\Rightarrow$ [iii]. If $\{P_n\}$ is UOT convergent to Q then $\{P_n\}$ is necessarily UI, since $\{P_n\}$ is bounded and $\|(I - T)P_n\| = \|(I - T)(P_n - Q)\| \leqslant \|I - T\| \, \|P_n - Q\|$.

[vii] $\Rightarrow$ [v]. With Θ above bounded,

$$\|P_n - Q\| = \|(I - Q)P_n\| = \|\Theta(I - T)P_n\| \leqslant \|\Theta\| \, \|(I - T)P_n\|.$$

[vi] $\Rightarrow$ [vii] (Lin). If $\|P - Q\| = \|P(I - Q)\| < 1$ then $(I - P)|\mathfrak{R}(I - Q)$ is invertible in $\mathfrak{B}(\mathfrak{R}(I - Q))$; determine $\gamma \in \mathfrak{B}(\mathfrak{X})$ by $\gamma(I - P) = (I - P)\gamma = I - Q$, $\gamma Q = Q\gamma = 0$. With $P = f(T) \in \mathcal{P}_1$, define $g(z)$ by

$$g(z) = [f(z) - 1]/(z - 1);$$

this is analytic on an appropriate neighborhood of $\sigma(T)$, so $g = g(T)$ exists in $\mathfrak{B}(\mathfrak{X})$ and $g(I - T) = (I - T)g = I - P$, $gQ = Qg$. It is straightforward that $\gamma g = g\gamma$ is the previously discussed $\Theta \in \mathfrak{B}(\mathfrak{X})$. $\square$

Since $\sigma(T; \mathfrak{X}) = \sigma(T; \mathfrak{Q}) \cup \sigma(T; \mathfrak{R}(I - Q))$ and $\sigma(T; \mathfrak{Q}) \subset \{1\}$, we have $\sigma(T; \mathfrak{R}(I - Q)) \supset [\sigma(T) \setminus \{1\}]^{-}$; condition [viii] necessarily fails if 1 is an accumulation point of $\sigma(T)$. Suppose $1 \in \sigma(T)$ is isolated in $\sigma(T)$, and let $E = (2\pi i)^{-1}\int_C R_\lambda(T)\, d\lambda$ be the corresponding spectral projection; the contour is a small circle in $\rho(T)$ with 1 interior and $\sigma(T) \setminus \{1\}$ exterior. From $R_\lambda(T)Q = QR_\lambda(T) = Q/(\lambda - 1)$ follows $EQ = QE = Q$. Let $\Delta = E - Q$ denote the complementary part: $\Delta^2 = \Delta$, $\Delta E = E\Delta = \Delta$, $\Delta Q = Q\Delta = 0$. If $\Delta \neq 0$ define $N \in \mathfrak{B}(\mathfrak{X})$ by $T\Delta = \Delta + N$. Then $N\Delta = \Delta N = N$ and $\sigma(N; \mathfrak{X}) = \{0\}$. Since N is quasinilpotent,

$$R_\lambda(T)\Delta = \frac{\Delta}{(\lambda - 1)I - N} = \frac{\Delta}{\lambda - 1} + \frac{N}{(\lambda - 1)^2} + \frac{N^2}{(\lambda - 1)^3} + \cdots, \qquad \lambda \neq 1,$$

is entire in $1/(\lambda - 1)$. Furthermore, $Nx \neq 0$ if $0 \neq x \in \mathfrak{R}(\Delta)$; otherwise $T(\Delta x) = (\Delta x)$ would imply $Qx = x$, contradicting $Q\Delta = 0$. It is obvious from this that $N^j x \neq 0$ if $0 \neq x \in \mathfrak{R}(\Delta)$ for each $j \geqslant 0$. It follows that $R_\lambda(T)$ has an essential singularity at $\lambda = 1$ if $\Delta \neq 0$, and Sine's problem is: characterize the case $\Delta = 0$.

3. A class of counterexamples. We show that condition [viii] of Theorem 1 fails in an interesting class of examples. Let X be a noncompact locally compact Hausdorff space, and let $C_0(X)$ be the continuous complex functions vanishing at infinity, with the usual supremum norm. We identify the bounded Radon measures $\mathfrak{M}(X)$ as the Banach space conjugate of $C_0(X)$, and denote by $\mathfrak{M}^+(X)$ the elements $\nu \geqslant 0$.

Let T: $C_0(X) \to C_0(X)$ be a Feller operator on $C_0(X)$. By this we mean

$$(Tf)(x) = \int f(x')t_x\,(dx'), \qquad x \in X, f \in C_0(X),$$

where the representing kernel t: $X \to \mathfrak{M}(X)$ is weakly* continuous and has the properties $t_x \geqslant 0$ and $t_x(X) = 1$, $x \in X$, and $\lim_{x\to\infty} t_x(K) = 0$ for any compact set $K \subset X$. The adjoint T^*: $\mathfrak{M}(X) \to \mathfrak{M}(X)$ has representation

$$(T^*\nu)(A) = \int_X t_x(A)\nu\,(dx), \qquad \text{Borel } A \subset X, \nu \in \mathfrak{M}(X).$$

We note that $\|\nu\| = \nu(X)$ when $\nu \in \mathfrak{M}^+(X)$, so $\|T^*\nu\| = \|\nu\|$, $\nu \in \mathfrak{M}^+(X)$; this implies $\|T\| = 1$.

We assume that the action of T is either null recurrent or transient:

$$\limsup_{N\to\infty}(1/N)\sum_{n=0}^{N-1}(T^{*n}\nu)(K)=0, \tag{1}$$

$$\text{each compact subset } K\subset X,\ \nu\in M^+(X).$$

To be a little more explicit, each compact subset K of the state space X is either null recurrent or transient for the discrete parameter Markov process which has $\nu/\|\nu\|$ as initial probability and t as transition probability.

THEOREM 2. *Let T: $C_0(X)\to C_0(X)$ be a null recurrent or transient Feller operator on noncompact locally compact Hausdorff space X. Then UI holds, and Q exists with $Q=0$. However, $\sigma(T)\ni 1$, and the resolvent $R_\lambda(T)$ cannot have a simple pole at $\lambda=1$.*

PROOF. Each T^j: $C_0(X)\to C_0(X)$ is a Feller operator, with representing kernel $t^{(j)}$, $j\geqslant 0$. Since $\|T^j\|=1$, the usual $A_n=(1/n)\Sigma_{j=0}^{n-1}T^j$, $n\geqslant 1$, constitute a UI sequence. Let us first prove $\mathfrak{Q}=0$. The modulus of any $f\in C_0(X)$ attains $\|f\|$ on X, since f vanishes at infinity. Suppose $0\neq f\in\mathfrak{Q}$, let $x_0\in X$ be such that $|f(x_0)|=\|f\|$, and define a nonempty compact set K by $K=\{x\in X: f(x)=f(x_0)\}$. From $T^jf=f$ and $t^{(j)}\geqslant 0$, $t^{(j)}(X)=1$, we obtain $t_x^{(j)}(K)=1$, $x\in K, j\geqslant 0$. This contradicts (1), proving $\mathfrak{Q}=0$.

We show next that $\mathfrak{Q}^*=0$. If $0\neq\nu\in\mathfrak{Q}^*$, the real and imaginary parts of ν are separately invariant, since T^* is real. Thus we may assume that $0\neq\nu\in\mathfrak{Q}^*$ is real. Then $0\neq|\nu|\in\mathfrak{Q}^*$, by the well-known argument $T^*|\nu|\geqslant|\nu|$, $\|T^*|\nu|\,\|=\|\,|\nu|\,\|$. Suppose a compact set K is such that $|\nu|(K)>0$. Then $(T^{*n}|\nu|)(K)=|\nu|(K)>0$ contradicts (1), showing $\mathfrak{Q}^*=0$. By the Sine Ergodic Theorem, Q exists; moreover, $Q=0$.

It remains to show that $\sigma(T)\ni 1$. The element $\mathcal{I}$ of the conjugate Banach space $\mathfrak{M}^*(X)$ is determined by

$$(\nu,\mathcal{I})=\nu(X),\qquad \nu\in\mathfrak{M}(X).$$

From $(\nu,\mathcal{I})=\|\nu\|$, $\nu\in\mathfrak{M}^+(X)$, follows $\mathcal{I}\neq 0$. The condition $t_x(X)=1$, $x\in X$, implies $(T^*\nu,\mathcal{I})=(\nu,\mathcal{I})$, $\nu\in\mathfrak{M}(X)$, and hence $T^{**}\mathcal{I}=\mathcal{I}$, where T^{**}: $\mathfrak{M}^*(X)\to\mathfrak{M}^*(X)$ is the adjoint of T^*. Since $\mathcal{I}\neq 0$, it belongs to the point spectrum at $\lambda=1$, i.e., $\sigma(T^{**})\ni 1$. But $\sigma(T^{**})=\sigma(T)$ by [1, Lemma VII.3.7]. $\square$

Theorem 2 is applicable in a more general setting. We are not concerned with boundary theory here, so we present matters without proof. We confine our remarks to the transient case. Suppose the space X above is open and dense in a compact Hausdorff space Y, and that the boundary $M=Y\setminus X$ is regular; the following conditions are supposed to be satisfied. First, the property

$$\lim_{\substack{x\to y\\ x\in X}} t_x=\delta_y\quad\text{weakly* in } \mathfrak{M}(Y),\qquad y\in M,$$

where δ_y is the unit point measure at y, extends the given operator T: $C_0(X)\to C_0(X)$ to a Feller operator $\check{T}$: $C(Y)\to C(Y)$. Also, $\check{Q}$ exists for $\check{T}$, the Choquet boundary of $\check{\mathfrak{Q}}$ is M, and $\check{\mathfrak{Q}}$ is isometrically isomorphic to $C(M)$. Then $\mathfrak{R}(I - \check{Q}) = \{f \in C(Y): f(M) = 0\}$, and this is isometrically isomorphic to $C_0(X)$. By Theorem 2, $\sigma(\check{T}; \mathfrak{R}(I - \check{Q})) \ni 1$, so the uniform ergodic theorem fails for any such $\check{T}$.

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