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A NOTE ON A LEMMA OF SHELAH CONCERNING STATIONARY SETS

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ABSTRACT. Let κ be an infinite cardinal, let I be a nonprincipal ideal on κ and let $I^+ = \{X \subseteq \kappa: X \notin I\}$. $S(I)$ is the following property of ideals: for every $A \in I^+$ and every pair of functions f, g from A into κ such that, for every $\alpha \in A$, $f(a) \neq g(\alpha)$, there exists a set $B \subseteq A$ with $B \in I^+$ such that $f''B \cap g''B = \varnothing$. We prove that $S(I)$ holds for every weakly selective ideal I on any infinite cardinal κ (including $\kappa = \omega$), and that $S(I)$ holds for every κ-complete ideal on κ iff κ is not strongly inaccessible.

Let κ be an infinite cardinal. A (proper) *ideal* on κ is a collection I of subsets of κ such that $\kappa \notin I$ and whenever $X, Y \in I$ and $Z \subseteq X \cup Y$, then $Z \in I$. If I is an ideal on κ then I^+ denotes the sets of "positive I-measure"; i.e. $I^+ = \{X \subseteq \kappa: X \notin I\}$. $S(I)$ is the following property of ideals: for every $A \in I^+$ and every pair of functions f, g from A into κ such that, for every $\alpha \in A$, $f(\alpha) \neq g(\alpha)$, there exists a set $B \subseteq A$ with $B \in I^+$ such that $f''B \cap g''B = \varnothing$. Shelah's lemma **[EM]** is the assertion $S(NS_\kappa)$, where NS_κ is the ideal of nonstationary subsets of the regular uncountable cardinal κ. The following result will provide a short proof of a generalization of Shelah's lemma.

THEOREM 1. *Let $S'(I)$ denote the weaker version of $S(I)$ obtained by considering only functions f and g that are one-to-one. Then $S'(I)$ holds for every ideal I on every infinite cardinal κ (including $\kappa = \omega$).*

PROOF. Let G be the graph on A obtained by making α adjacent to β (where $\alpha < \beta$) iff $g(\alpha) = f(\beta)$. Then each point $B \in A$ is adjacent to at most one $\alpha < \beta$ (since otherwise we would have $f(\beta) = g(\alpha_1)$ and $f(\beta) = g(\alpha_2)$ contradicting the one-to-oneness of g). Thus each $\beta \in A$ gives rise to a unique decreasing path of finite length. Without loss of generality, assume that the set B' of points $\beta \in A$ having such a path of even length is of positive I-measure. Since B' is clearly an independent set in the graph G it follows that if we have $\alpha, \beta \in B'$ with $\alpha < \beta$ then $g(\alpha) \neq f(\beta)$. Now we simply repeat the procedure (starting with B') with the roles of f and g reversed. The set $B \subseteq B'$ of positive I-measure so obtained clearly has the property that $f''B \cap g''B = \varnothing$ as desired. $\square$

REMARK. It is worth noting that we really do not need both f and g to be one-to-one–just the "larger." That is, if we let $A_g = \{\alpha \in A: f(\alpha) < g(\alpha)\}$ and

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$A_f = \{\alpha \in A\colon g(\alpha) < f(\alpha)\}$ then either $A_f \in I^+$ or $A_g \in I^+$. If, for example, $A_g \in I^+$ then we can redo the second step in the above proof so as to appeal to this fact instead of the one-to-oneness of f as follows. Let G' be the graph on B' in which α is adjacent to β (where $\alpha < \beta$) iff $f(\alpha) = g(\beta)$. Then each α is adjacent to at most one $\beta > \alpha$ (since g is one-to-one). Notice also that if $\alpha < \beta$ and α is adjacent to β then $g(\alpha) > g(\beta)$; that is, $g(\beta) = f(\alpha) < g(\alpha)$. Hence each α gives rise to a unique increasing path of finite length and so we can proceed exactly as in the first part of the proof of Theorem 1.

Recall that an ideal I on κ is said to be *normal* if every regressive function f (i.e. $f(\alpha) < \alpha$ for $\alpha \neq 0$) defined on a set of positive I-measure is constant on a set of positive I-measure. (Fodor's theorem **[F]** asserts that NS_κ is normal if κ is a regular uncountable cardinal.) I is said to be *weakly selective* if every function defined on a set of positive I-measure is either constant on a set of positive I-measure or one-to-one on a set of positive I-measure. Weglorz first observed that every normal ideal I is weakly selective. (In fact, if I is normal, $A \in I^+$, $f\colon A \to \kappa$ and $f^{-1}(\{\alpha\}) \in I$ for every $\alpha < \kappa$, then the set $B = A - \{\inf(f^{-1}(\{\alpha\}))\colon \alpha < \kappa\}$ is in I as can be seen by considering the regressive function $h\colon B \to \kappa$ given by $h(\alpha) = \inf(f^{-1}(\{\alpha\}))$.) Even on uncountable cardinals there are lots of weakly selective ideals that are not normal (e.g. $\{X \subseteq \kappa^+\colon |X| < \kappa^+\}$; for more see **[BTW]**). With this much said, an easy consequence of Theorem 1 is the following.

COROLLARY. *$S(I)$ holds for every weakly selective ideal I on any infinite cardinal κ (including $\kappa = \omega$).*

Theorem 1 and its corollary suggest the possibility that perhaps $S(I)$ holds for *every* ideal I. This, however, is easily seen not to be the case. For example, if D is an ultrafilter on κ and I is the ideal on $\kappa \times \kappa$ dual to $D \times D$, then the projection functions show that $S(I)$ fails. These considerations also show that if κ is a measurable cardinal then there is a *κ-complete* ideal I (that is, one closed under unions of size less than κ) for which $S(I)$ fails. On the other hand, one can use Theorem 1 (and the remark following it) to show that if κ is an infinite *successor* cardinal then $S(I)$ holds for *every* κ-complete ideal on κ. Hence. if we momentarily agree to call κ *good* iff $S(I)$ holds for every κ-complete ideal on κ, then we have that successor cardinals are good and measurable cardinals are not. Our next result will fill the obvious gap (i.e., it will follow that κ is good iff κ is not strongly inaccessible).

THEOREM 2. *For infinite cardinals κ and μ, the following are equivalent*:

(i)

$$\kappa \to \left(\begin{matrix}\bullet\\ \bullet\\ \bullet\end{matrix}\right)^2_{<\mu};$$

i.e., for every $f\colon [\kappa]^2 \to \lambda$ where $\lambda < \mu$, there exists α, β, γ such that $\alpha < \beta < \gamma < \kappa$ and $f(\{\alpha, \beta\}) = f(\{\beta, \gamma\})$.

(ii) *$S(I)$ fails for some proper nonprincipal μ-complete ideal I on κ.*

PROOF. (i) $\to$ (ii). Assume that

$$\kappa \to \left(\begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix}\right)^2_{<\mu}$$

and let $A = \{(\alpha, \beta): \alpha < \beta < \kappa\}$. We will construct a κ-complete proper nonprincipal ideal I on the set A so that the projection functions π_1 and π_2 show that $S(I)$ fails; this clearly suffices.

Let $\mathcal{S} = \{X \subseteq A: \pi_1(X) \cap \pi_2(X) = 0\}$ and let I be the μ-complete ideal on A generated by $\mathcal{S}$ (i.e., $Y \in I$ iff $Y \subseteq \bigcup H$ for some $H \subseteq \mathcal{S}$ with $|H| < \mu$). Then I is clearly closed downward (i.e., $Y \subseteq X \in I \Rightarrow Y \in I$) and under unions of size less than μ. Moreover, every singleton subset $\{(\alpha, \beta)\}$ of A is in I (since $\alpha \neq \beta$). Hence, it remains only to show that I is proper.

Suppose not. Then $A = \bigcup \{A_\xi: \xi < \lambda\}$ for some $\lambda < \mu$ where we have $A_\xi \in \mathcal{S}$ for each $\xi < \lambda$. We can assume that the A_ξ's are pairwise disjoint. Define f: $[\kappa]^2 \to \lambda$ by $f(\{\alpha, \beta\}) = \xi$ iff $\alpha < \beta$ and $(\alpha, \beta) \in A_\xi$. Since $\lambda < \mu$ and

$$\kappa \to \left(\begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix}\right)^2_{<\mu}$$

we get some $\xi < \lambda$ and $\alpha < \beta < \gamma$ so that $f(\{\alpha, \beta\}) = \xi = f(\{\beta, \gamma\})$. But then $(\alpha, \beta) \in A_\xi$ and $(\beta, \gamma) \in A_\xi$ so $\beta \in \pi_1'' A_\xi \cap \pi_2'' A_\xi$. This contradicts the fact that $A_\xi \in \mathcal{S}$ and thus shows that I is proper.

(ii) $\to$ (i). Suppose that h: $[\kappa]^2 \to \lambda$ for some $\lambda < \mu$ and h shows that

$$\kappa \not\to \left(\begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix}\right)^2_{<\mu}.$$

Let I be a proper μ-complete ideal on κ and suppose that f, g: $A \to \kappa$ where $A \in I^+$ and $f(\alpha) \neq g(\alpha)$ for every $\alpha \in A$. For each $\xi < \lambda$ let A_ξ be given by

$$A_\xi = \{\alpha \in A: h(\{f(\alpha), g(\alpha)\}) = \xi\}.$$

Since I is μ-complete, $\lambda < \mu$ and $A \in I^+$ we get that $A_\xi \in I^+$ for some $\xi < \lambda$. Without loss of generality assume that $B \in I^+$ where $B = \{\alpha \in A_\xi: f(\alpha) < g(\alpha)\}$. Now, to complete the proof it suffices to show that $f''B \cap g''B = \varnothing$.

Suppose not, and choose $\alpha, \gamma \in B$ such that $f(\alpha) = g(\gamma) = \beta$. Then $f(\gamma) < g(\gamma) = \beta = f(\alpha) < g(\alpha)$ and so $f(\gamma) < \beta < g(\alpha)$. But $h(\{f(\gamma), \beta\}) = h(\{f(\gamma), g(\gamma)\}) = \xi = h(\{f(\alpha), g(\alpha)\}) = h(\{\beta, g(\alpha)\})$ and so the set $\{f(\gamma), \beta, g(\alpha)\}$ contradicts the fact that h shows

$$\kappa \not\to \left(\begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix}\right)^2_{<\mu}. \quad \square$$

COROLLARY. *For regular cardinals κ and μ, the following are equivalent*:

(i) $2^\lambda \geqslant \kappa$ *for some* $\lambda < \mu$.

(ii) $S(I)$ *holds for every μ-complete proper ideal I on κ.*

PROOF. (i) $\to$ (ii). Assume that $\lambda < \mu$ and $2^\lambda \geqslant \kappa$. By the previous theorem it suffices to show that

$$\kappa \not\to \left(\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}\right)^2_\lambda;$$

our argument here is only a slight (but necessary) modification of the standard example (due to Erdös and Rado [ER]) showing that $2^\lambda \not\to (3)^2_\lambda$. So let $h\colon \kappa \to {}^\lambda 2$ be one-to-one where ${}^\lambda 2$ denotes the set of all functions mapping λ to 2. Define $f\colon [\kappa]^2 \to \lambda \times 2$ as follows. If $\alpha < \beta$ then set $f(\{\alpha, \beta\}) = (\gamma, i)$ where

$$\gamma = \inf\{\xi < \lambda\colon h(\alpha)(\xi) \neq h(\beta)(\xi)\}$$

and $h(\alpha)(\gamma) = i$. Now, suppose for contradiction that $\alpha < \beta < \delta$ and $f(\{\alpha, \beta\}) = (\gamma, i) = f(\{\beta, \delta\})$. Without loss of generality, assume that $i = 0$. Then $h(\alpha)(\gamma) = 0$ and $h(\beta)(\gamma) = 1$ (since $f(\{\alpha, \beta\}) = (\gamma, 0)$). But then since $f(\{\beta, \delta\}) = (\gamma, 0)$ we have $h(\beta)(\gamma) = 0$; contradiction.

(ii) $\to$ (i). The Erdös-Rado Theorem [ER] asserts that $(2^\lambda)^+ \to (\lambda^+)^2_\lambda$; it follows trivially from this that if $\kappa > 2^\lambda$ for every $\lambda < \mu$ then

$$\kappa \to \left(\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}\right)^2_{<\mu}.$$

The desired result thus follows from the previous theorem. $\square$

REMARK. A consequence of the above is that if $\kappa = \sup\{(2^\lambda)^+\colon \lambda < \kappa\}$ and κ is regular, then $S(I)$ fails for some proper nonprincipal μ-complete *uniform* ideal I on κ. (To say that I is uniform means that $\{X \subseteq \kappa\colon |X| < \kappa\} \subseteq I$.)

COROLLARY. *$S(I)$ holds for every κ-complete proper nonprincipal ideal I on κ iff κ is not strongly inaccessible.*

We conclude with an easy application of the corollary to Theorem 1. An ultrafilter $\mathcal{U}$ on κ is said to be Ramsey if every function $f\colon \kappa \to \kappa$ is either constant or a set in $\mathcal{U}$ or one-to-one on a set in $\mathcal{U}$. If $\mathcal{U}$ is an ultrafilter on κ and A is a set then a subset X of $A^\kappa/\mathcal{U}$ is called *standard* if there is a $B \subseteq A$ such that $X = B^\kappa/\mathcal{U}$. We claim that if $\mathcal{U}$ is a Ramsey ultrafilter on κ, then any two elements of $A^\kappa/\mathcal{U}$ can be separated by a standard set. That is, if $[f], [g] \in A^\kappa/\mathcal{U}$ and $[f] \neq [g]$, then the corollary to Theorem 1 yields a set $X \in \mathcal{U}$ so that $f''X \cap g''X = \varnothing$. But now if $B = f''X$, then $[f] \in B^\kappa/\mathcal{U}$ and $[g] \notin B^\kappa/\mathcal{U}$. This application has consequences for certain problems involving cardinalities of ultrapowers; these will appear elsewhere.

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