PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, December 1981

A REFINEMENT OF CANTOR'S THEOREM

G. KIRMAYER

ABSTRACT. It is shown that there is no surjection from the one-element subsets of a set containing an infinite co-infinite set to the infinite co-infinite subsets of that set. It is also shown that there is no surjection from the one-element subsets of an infinite set to the infinite subsets of that set. The proof can be formalized in a subtheory of both Zermelo Set Theory and New Foundations (and thus makes no use of the Axiom of Choice).

We will proceed informally. It should be clear that what follows can be formalized in the subtheory of both Zermelo Set Theory and New Foundations which has as axioms Extensionality, Pairing, Power Set, Union and Aussonderung for stratified formulas.

Let P_1X denote the set of one-element subsets of X.

Let P_IX denote the set of infinite subsets of X.

Let $P_{I,C}X$ denote the set of inifinite co-infinite subsets of X (i.e. the set of those $Y \in P_IX$ such that $X - Y$ is infinite).

THEOREM. (a) *Suppose X has an infinite, co-infinite subset. Then there is no surjective function from P_1X to $P_{I,C}X$.*

(b) *Let X be an infinite set. Then there is no surjective function from P_1X to P_IX.*

PROOF. (a) Suppose f is a surjective function from P_1X to $P_{I,C}X$. There is a set S such that $\forall x\ (x \in S \equiv x \in X \wedge x \notin f\{x\})$. If S is an infinite, co-infinite subset of X and $f\{s\} = S$ then $s \in S \equiv s \notin S$. Therefore S is finite or $X - S$ is finite. Let $g = f$ if S is finite and let g be defined by $g\{x\} = X - f\{x\}$ if $X - S$ is finite. Let $R = S$ if S is finite and let $R = X - S$ if $X - S$ is finite. Then g is a surjective function from P_1X to $P_{I,C}X$, $\forall x\ (x \in R \equiv x \in X \wedge x \notin g\{x\})$, and R is finite.

Let $a \in X - R$. There is a set T such that $\forall x\ (x \in T \equiv x \in X \wedge a \in g\{x\})$. $(R \cup T) - \{a\}$ is an infinite, co-finite subset of X. Suppose $g\{b\} = (R \cup T) - \{a\}$. If $b \in R$ then $b \notin (R \cup T) - \{a\}$ and thus $b \notin R$. Therefore $b \notin R$ and $b \in (R \cup T) - \{a\}$. But $b \notin T$ because $a \notin g\{b\}$. Therefore there can be no surjective function from P_1X to $P_{I,C}X$.

(b) Suppose f is a surjective function from P_1X to P_IX. Let $a \in X$. There is a set Y such that $\forall y\ (y \equiv Y \in y \in X \wedge a \in f\{y\})$. Y is an infinite, co-infinite subset of X. Therefore, by (a), there is no surjective function from P_1X to $P_{I,C}X$. Thus there can be no surjective function from P_1X to P_IX. This contradiction establishes the result.

Current address: Department of Mathematics, Wichita State University, Wichita, Kansas 67208

Received by the editors February 4, 1981.
1980 *Mathematics Subject Classification*. Primary 03E04.