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SELECTION AND REPRESENTATION THEOREMS FOR σ-COMPACT VALUED MULTIFUNCTIONS

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ABSTRACT. In this paper we give two applications of results of Shchegolkov and Saint-Raymond on Borel sets with σ-compact sections. First we give a sufficient condition under which a partition of a Polish space into σ-compact sets admits a Borel cross-section. Then a representation theorem for σ-compact valued multifunctions, expressing them as unions of continuously indexed Borel graphs, is proved.

1. Introduction. The past few years have seen much progress in selection theory. It has been shown that multifunctions whose values are topological nice subsets of a Polish space (e.g. countable, closed, G_δ, nonmeager) admit a measurable selector. More generally, they can be represented as the union of graphs of measurable selectors which are themselves nicely indexed. For details we refer the reader to Wagner [7, 8].

In this paper we study multifunctions whose values are σ-compact subsets of a Polish space. Some positive results on such multifunctions were proved in the late 30's and early 40's by Kunugui, Novikov, Arsenin and Shchegolkov. Our main results can be stated as follows.

THEOREM 1.1. *Let $\underline{\underline{Q}}$ be a partition of a Polish space T into σ-compact sets. Then the following conditions are equivalent*:

(i) *The σ-field $\underline{\underline{A}}(\underline{\underline{Q}})$ of all Borel sets in T which are unions of elements of $\underline{\underline{Q}}$ is countably generated.*

(ii) *The equivalence relation $\underline{\underline{R}}(\underline{\underline{Q}})$ induced by $\underline{\underline{Q}}$ belongs to $\underline{\underline{A}}(\underline{\underline{Q}}) \otimes \underline{\underline{B}}_T$, where $\underline{\underline{B}}_T$ is the Borel σ-field of T.*

Moreover if (i) *or* (ii) *holds then $\underline{\underline{Q}}$ admits a Borel cross-section.*

THEOREM 1.2. *Let T be an analytic space and X a Polish space. If F: $T \to X$ is a σ-compact valued multifunction such that* graph$(F) \in \underline{\underline{B}}_{T\times X}$ *then there is a map f: $T \times (\omega \times C) \to X$ such that*

(i) *for each $t \in T$, $f(t, \cdot)$ is a continuous function from $\omega \times C$ onto $F(t)$, and*

(ii) *for $m \in \omega$ and $\alpha \in C$, the map $t \to f(t, m, \alpha)$ is a Borel measurable selector for F,*

where ω is the space of natural numbers and C denotes the Cantor space $\{0, 1\}^\omega$.

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The paper is organised as follows. In §2 we introduce the basic definitions and notation and record some preliminary results. A proof of Theorem 1.1 is given in §3. Theorem 1.2 is established in §4.

The results of this paper are included in my thesis. I express my indebtedness to Professor A. Maitra for his guidance.

2. Preliminaries. A second countable completely metrizable topological space is called a *Polish space*. We denote the set of natural numbers by ω, the Cantor set $\{0, 1\}^\omega$ by C and the Baire space ω^ω of infinite sequences of natural numbers will be denoted by Σ. A metrizable space T is called *analytic* if it is a continuous image of Σ. If X is a Polish space then $A \subset X$ is called *coanalytic* if $X \setminus A$ is analytic. If T is a metrizable space, $\underline{\underline{B}}_T$ will stand for the Borel σ-field of T, i.e. $\underline{\underline{B}}_T$ is the σ-field generated by open sets in T. For results in descriptive set theory we refer the reader to Kuratowski [2]. The set of finite sequences of natural numbers including the empty sequence $\varnothing$ is denoted by seq. For any $s \in \mathrm{Seq}$, $\mathrm{lh}(s)$ denotes the length of s and if $i < \mathrm{lh}(s)$, s_i will denote the ith coordinate of s. Moreover, if $k \in \omega$ and $\alpha \in \omega^\omega$ then sk denotes the catenation of s and k and $\alpha|k = (\alpha(0), \ldots, \alpha(k-1))$. In particular, $\alpha|0 = e$.

Let $(T, \underline{\underline{A}})$ and $(S, \underline{\underline{B}})$ be measurable spaces. Then $\underline{\underline{A}} \otimes \underline{\underline{B}}$ denotes the product of the σ-fields $\underline{\underline{A}}$ and $\underline{\underline{B}}$. An $\underline{\underline{A}}$-*atom* is a nonempty set $A \in \underline{\underline{A}}$ such that $\varnothing \neq B \subseteq A$ and $B \in \underline{\underline{A}}$ imply $B = A$. If $E \subseteq T \times S$ and $t \in T$ then E^t denotes the set $\{s \in S: (t, s) \in E\}$ and is called the *section* of E at t. We denote the projection map from $T \times S$ onto T by $\pi_T^{T \times S}$ or simply by π_T if there is no ambiguity. We say that the σ-field $\underline{\underline{A}}$ is *countably generated* if there exist subsets $A_1, A_2, \ldots,$ of T such that $\underline{\underline{A}}$ is generated by $\{A_n\}$. If $A_1, A_2, \ldots$ are subsets of T then by the *characteristic function* of the sequence $\{A_n\}$ is meant the function f: $T \to [0, 1]$, defined by

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{3^n} 1_{A_n}(t),$$

where 1_{A_n} denotes the indicator of the set A_n. If $\underline{\underline{A}}$ is a σ-field generated by the sequence $\{A_n\}$ and if f: $T \to [0, 1]$ is the characteristic function of $\{A_n\}$ then $\underline{\underline{A}}$: $f^{-1}(\underline{\underline{B}}_Z)$ where $Z = f(T)$. This fact will be used in the sequel without explicit mention. If T and S are nonempty sets and $A \subseteq T \times S$, a set $B \subseteq A$ is called a *uniformization* of A if $\pi_T(B) = \pi_T(A)$ and B^t is at most a singleton for every $t \in T$.

Let $(T, \underline{\underline{A}})$ be a measurable space and X be a Polish space. A *multifunction* F: $T \to X$ is a map whose domain is T and whose values are nonempty subsets of X. For $E \subseteq X$, we denote the set $\{t \in T: F(t) \cap E \neq \varnothing\}$ by $F^{-1}(E)$. We say that F is $\underline{\underline{A}}$-*measurable* if $F^{-1}(U) \in \underline{\underline{A}}$ for all open sets U in X. The set $\{(t, x) \in T \times X: x \in F(t)\}$ is called the *graph* of F and will be denoted by graph(F). A point map f: $T \to X$ is called a *selector* for F if $f(t) \in F(t)$ for all $t \in T$.

A collection $\underline{\underline{Q}}$ of pairwise disjoint, nonempty subsets of X whose union is X is called a *partition* of X. For $E \subset X$, $E^{*\underline{\underline{Q}}}$ or simply E^* denotes the set $\bigcup\{A \in \underline{\underline{Q}}: A \cap E \neq \varnothing\}$ and is called $\underline{\underline{Q}}$-*saturation* of E. We say that E is $\underline{\underline{Q}}$-*saturated* if $E = E^*$. If X is a Polish space then $\underline{\underline{Q}}$ is *measurable* if U^* is Borel for all open sets U in X. Also, $\underline{\underline{A}}(\underline{\underline{Q}})$ will denote the σ-field of all $\underline{\underline{Q}}$-saturated Borel sets in X. We shall denote the equivalence relation induced by $\underline{\underline{Q}}$ by $\underline{\underline{R}}(\underline{\underline{Q}})$. A subset S of X is called a *cross-section* of $\underline{\underline{Q}}$ if $S \cap E$ is a singleton for all $E \in \underline{\underline{Q}}$.

Let (Z, d) be a metric space, ε a positive real number, $z \in Z$ and $A \subseteq Z$. Then $S_\varepsilon(z)$ will denote the open ball $\{z' \in Z: d(z, z') < \varepsilon\}$ and $\mathrm{cl}(A)$ will denote the topological closure of A.

We shall now state some lemmas which will be used in the sequel.

LEMMA 2.1. *Let $(T, \underline{\underline{A}})$ be a measurable space, X a Polish space and $F: T \to X$ a closed valued, $\underline{\underline{A}}$-measurable multifunction. If $f: T \to X$ is an $\underline{\underline{A}}$-measurable selector for F and ε a positive real number then the multifunction $H: T \to X$ defined by*

$$H(t) = \mathrm{cl}(F(t) \cap S_\varepsilon(f(t))), \qquad t \in T,$$

is $\underline{\underline{A}}$-measurable.

A proof of this lemma can be found in **[6]**.

LEMMA 2.2. *Let T be an analytic space and X a Polish space. If $B \in \underline{\underline{B}}_{T \times X}$ and B^t is compact for all $t \in T$ then $\pi_T(B) \in \underline{\underline{B}}_T$.*

PROOF. Embed T in a Polish space Z and let $E \in \underline{\underline{B}}_{Z \times X}$ such that $B = E \cap (T \times X)$. Suppose $P = \{z \in Z: E^z$ is nonempty and compact$\}$. Then P is coanalytic **[5**, p. 224] and $\pi_T(B) \subset P$. We get a Borel set R in Z such that $\pi_T(B) \subset R \subset P$ **[2**, p. 485]. Then $R \cap T = \pi_T(B)$. Hence $\pi_T(B)$ is Borel in T.

We shall use the following deep facts about Borel sets with σ-compact sections in the sequel.

THEOREM 2.3. *Let T be an analytic and X a Polish space. If B is a Borel subset of $T \times X$ such that B^t is σ-compact for every $t \in T$, then*

(a) *$\pi_T(B)$ is Borel in T,*

(b) *B can be uniformized by a Borel set in T, and*

(c) *there exist Borel sets $B_0, B_1, B_2, \ldots$ in $T \times X$ such that $B = \bigcup_n B_n$ and B_n^t is compact for all n and t.*

Part (a) of this theorem was proved independently by Arsenin and Kunugui, (b) is the uniformization theorem of Shchegolkov, and part (c) is a very difficult theorem of Saint-Raymond **[4]**.

3. Proof of Theorem 1.1. (i) $\Rightarrow$ (ii). Let $A_1, A_2, \ldots$ generate $\underline{\underline{A}}(\underline{\underline{Q}})$ and suppose f is the characteristic function of $\{A_n\}$. Then f is $\underline{\underline{A}}(\underline{\underline{Q}})$-measurable and $\underline{\underline{R}}(\underline{\underline{Q}}) = \{(t, t') \in T \times T: f(t) = f(t')\}$. Therefore, $\underline{\underline{R}}(\underline{\underline{Q}}) \in \underline{\underline{A}}(\underline{\underline{Q}}) \otimes \underline{\underline{A}}(\underline{\underline{Q}}) \subseteq \underline{\underline{A}}(\underline{\underline{Q}}) \otimes \underline{\underline{B}}_T$.

(ii) $\Rightarrow$ (i). Let $A_1, A_2, \ldots \in \underline{\underline{A}}(\underline{\underline{Q}})$ and $B_1, B_2, \ldots \in \underline{\underline{B}}_T$ be such that $\underline{\underline{R}}(\underline{\underline{Q}})$ belongs to the σ-field generated by $A_1 \times B_1, A_2 \times B_2, \ldots$. Let $\underline{\underline{M}}$ be the σ-field generated by $A_1, A_2, \ldots$. Then $\underline{\underline{M}}$ is a countably generated sub-σ-field of $\underline{\underline{A}}(\underline{\underline{Q}}) \subseteq \underline{\underline{B}}_T$ and $\underline{\underline{M}}$ and $\underline{\underline{A}}(\underline{\underline{Q}})$ have the same set of atoms. Hence by a theorem of Blackwell **[1**, Theorem 3] $\underline{\underline{M}} = \underline{\underline{A}}(\underline{\underline{Q}})$. This proves (i).

To prove the last part of the theorem let $A_1, A_2, \ldots$ be a countable generator of $\underline{\underline{A}}(\underline{\underline{Q}})$ and, suppose f is the characteristic function of the sequence $\{A_n\}$. Let

$P \subseteq [0, 1]$ be the range of f. Then P is Borel (Theorem 2.3(a)). Define a multifunction $H: P \to T$ by

$$H(p) = \{t \in T: f(t) = p\}, \qquad p \in P.$$

Then $H(p)$ is σ-compact for all $p \in P$ and graph$(H) = \underline{\underline{B}}_{P \times T}$. Therefore, by the uniformization theorem of Shchegolkov, there is a Borel selector $h: P \to T$ for H. Put $S = \{t \in T: h(f(t)) = t\}$. It is easily seen that S is a Borel cross-section for Q.

REMARK. Consider the partition of the real line $\mathbf{R}$ given by "x is equivalent to y if $x - y$ is a rational number". Members of this partition are all countable and the saturation of every open set is open. But it is well known that this does not admit even a Lebesgue measurable cross-section. This shows that the hypothesis of the theorem cannot be relaxed.

4. Representation theorems. We first prove a representation theorem for compact valued multifunctions.

LEMMA 4.1. *If $(T, \underline{\underline{A}})$ is a measurable space, X a compact metric space and ε a positive real number then there is a positive integer n such that for every compact valued, $\underline{\underline{A}}$-measurable multifunction F: $T \to X$ there exist $\underline{\underline{A}}$-measurable selectors $f_0, f_1, \ldots, f_n$ for F such that $f_0(t), \ldots, f_n(t)$ is an ε net in $F(t)$ for all t.*

PROOF. Let n be a positive integer that there exist open sets $W_0, W_1, \ldots, W_n$ in X of diameters less than ε which cover X. Fix a compact valued, $\underline{\underline{A}}$-measurable multifunction F: $T \to X$. Fix an $\underline{\underline{A}}$-measurable selector g: $T \to X$ for F. Define $T_i = F^{-1}(W_i)$, $0 \leq i \leq n$. Then $T_i \in \underline{\underline{A}}$ and $\bigcup_{i=0}^{n} T_i = T$.

For any nonnegative integer i, less than or equal to n, define a multifunction F_i: $T_i \to W_i$ by

$$F_i(t) = F(t) \cap W_i, \qquad t \in T_i.$$

Then F_i is $\underline{\underline{A}}$-measurable and closed valued. Hence there is an $\underline{\underline{A}}$-measurable selector g_i: $T_i \to W_i$ for F_i [3]. Now define f_i: $T \to X$ by

$$\begin{aligned} f_i(t) &= g_i(t) \quad \text{if } t \in T_i, \\ &= g(t) \quad \text{if } t \in T - T_i. \end{aligned}$$

Clearly $f_0, \ldots, f_n$ have the required properties.

THEOREM 4.2. *Let $(T, \underline{\underline{A}})$ be a measurable space and X a Polish space. Suppose F: $T \to X$ is a compact valued, $\underline{\underline{A}}$-measurable multifunction. Then there is a point map f: $T \times C \to X$ such that*

(i) *for all $t \in T$, $f(t, \cdot)$ is a continuous map from C onto $F(t)$, and*

(ii) *for all $\alpha \in C$, the function $f(\cdot, \alpha)$ is $\underline{\underline{A}}$-measurable.*

PROOF. Since any Polish space can be embedded in a compact metric space, without any loss of generality, we assume that X is a compact metric space.

We show that there are positive integers $n_0, n_1, n_2, \ldots$ and for each $s \in$ Seq with $s_i \leq n_i$, $i < \text{lh}(s)$, an $\underline{\underline{A}}$-measurable selector g_s: $T \to X$ for F such that for all $t \in T$, $\{g_{si}(t): i \leq n_k\}$ is a $2^{-(k+1)}$-net in $\text{cl}(F(t) \cap S_{2^{-k}}(g_s(t)))$ where $k = \text{lh}(s)$. To show this we proceed by induction.

Let g_e be any $\underline{\underline{A}}$-measurable selector for F. By Lemma 4.1 we get a positive integer n_0 and $\underline{\underline{A}}$-measurable selectors $g_0, g_1, \ldots, g_{n_0}$ for F such that $g_0(t), \ldots, g_{n_0}(t)$ is a $1/2$ net in $F(t)$ for all t. Suppose for some positive integer k, positive integers $n_0, \ldots, n_{k-1}$ and functions g_s for $s \in \mathrm{Seq}$ with $\mathrm{lh}(s) < k$ and $s_i \leqslant n_i$ for $0 \leqslant i < \mathrm{lh}(s)$ satisfying the above conditions have been defined. Fix an $s \in \mathrm{Seq}$ such that $\mathrm{lh}(s) = k$ and $s_i \leqslant n_i$ for $0 \leqslant i < k$. Define a multifunction F_s: $T \to X$ by

$$F_s(t) = \mathrm{cl}\big(F(t) \cap S_{2^{-k}}(g_s(t))\big), \qquad t \in T.$$

Then by Lemma 2.1, F_s is $\underline{\underline{A}}$-measurable. By Lemma 4.1 we get a positive integer n_k (independent of s) and $\underline{\underline{A}}$-measurable selectors $g_{s0}, g_{s1}, \ldots, g_{sn_k}$ for F_s such that, for all t, $g_{s0}(t), \ldots, g_{sn_k}(t)$ is a $2^{-(k+1)}$-net in $F_s(t)$. It is easy to check that positive integers $n_0, n_1, \ldots$ and the function g_s: $T \to X$ thus defined have desired properties.

Put $Y = \times_{i=0}^{\infty} \{0, 1, \ldots, n_i\}$ and give it the product of discrete topologies. Take a $\delta \in Y$ and $t \in T$. Then $\{g_{\delta|k}(t)\colon k \in \omega\}$ is a Cauchy sequence in X. Define $f(t, \delta) = \lim_{k\to\infty} g_{\delta|k}(t)$. It is easily checked that f satisfies (i) and (ii). Since Y is homeomorphic to C the proof is complete.

Proof of Theorem 1.2. Since any Polish space can be embedded in a compact metric space as a G_δ, without any loss of generality, we assume that X is a compact metric space.

Let $G = \mathrm{graph}(F)$ and $G_0, G_1, G_2, \ldots$ be a sequence of Borel sets in $T \times X$ such that G_n^t is compact for all n and t and $G = \bigcup_n G_n$ (Theorem 2.3(c)).

Fix an $n \in \omega$. Let $m_0, m_1, \ldots$ be an enumeration of ω such that $m_0 = n$. Suppose

$$\begin{aligned} T_i^n &= \pi_T(G_{m_i}) \quad \text{if } i = 0, \\ &= \pi_T(G_{m_i}) - \bigcup_{j<i} \pi_T(G_{m_j}) \quad \text{if } i > 0. \end{aligned}$$

By Lemma 2.2, $T_i^n \in \underline{\underline{B}}_T$, $i \neq j \to T_i^n \cap T_j^n = \varnothing$ and $\bigcup_{i=0}^{\infty} T_i^n = T$. Define a multifunction F_n: $T \to X$ by $F_n(t) = G_{m_i}^t$ if $t \in T_i^n$. Then F_n is compact valued and, by Lemma 2.2, $\underline{\underline{B}}_T$-measurable. By Theorem 4.2 we get a map f_n: $T \times C \to X$ such that, for all $t \in T$, $f_n(t, \cdot)$ is continuous and onto $F_n(t)$ and, for all $\alpha \in C$, the map $f_n(\cdot, \alpha)$ is $\underline{\underline{B}}_T$-measurable. Define f: $T \times (\omega \times C) \to X$ by

$$f(t, m, \alpha) = f_m(t, \alpha), \qquad t \in T, m \in \omega, \alpha \in C.$$

It is easily checked that f has the desired properties.

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