PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, February 1982

STRICT LOCAL RINGS

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ABSTRACT. In this paper we introduce the notion of a strict local ring. A local Cohen-Macaulay ring (B, m) is called strict if, whenever a local ring (A, n) specializes by a regular sequence to B, then the associated graded ring $\mathrm{gr}_n(A)$ is Cohen-Macaulay. We show that an artinian graded algebra B is strict if for the graded cotangent module we have $T^1(B/k, B)_\nu = 0$ for $\nu < -1$. Various examples are considered where this condition holds. In particular, with this method we reprove a result of J. Sally [6].

The purpose of this paper is to extend results of J. Sally who showed that for a certain class of local Cohen-Macaulay rings (B, m) the graded ring $\mathrm{gr}_m(B)$ is again Cohen-Macaulay. A typical result of this type proved by J. Sally is the following [6]: Let (B, m) be a d-dimensional local Cohen-Macaulay ring with multiplicity e and embedding dimension $e - d + 1$. Then $\mathrm{gr}_m(R)$ is Cohen-Macaulay. For the proof of the theorem one may assume that B/m is infinite. Then there exist elements $x_1, \ldots, x_d \in m$ such that $\overline{m}^2 = 0$ for the maximal ideal $\overline{m}$ of $\overline{B} = B/(x_1, \ldots, x_d)$. Thus all local rings with fixed embedding dimension equal to $e - d + 1$ specalize to the same artinian local ring. Conversely all local rings (B, m) specializing to $(\overline{B}, \overline{m})$ with $\overline{m}^2 = 0$ have embedding dimension $e + d - 1$ and hence have the property that $\mathrm{gr}_m(B)$ is Cohen-Macaulay.

It is thus natural to make the following definition: A local Cohen-Macaulay ring B is said to be *strict* if it has the following property: Given a local ring (A, n) and a regular A-sequence $\underline{x}$ such that $B \simeq A/(\underline{x})A$, then $\mathrm{gr}_n(A)$ is Cohen-Macaulay.

Although we have formulated this definition in great generality we will restrict ourselves to studying only the most important case of strict artinian local rings. The nonartinian case is treated very similarly.

For technical reasons we consider only analytic k-algebras $k[|X_1 \ldots X_n|]/I$, where k is an infinite field. This is for the questions under consideration not very restrictive, since one may always assume that the local ring is complete and has infinite residue class field.

In [3, Theorem 2] we considered already certain classes of strict artinian algebras. There we showed that $B = k[|X_1 \ldots X_n|]/I$ is strict, if I is generated by forms of degree 2 and no surjection $I/I^2 \to B$ exists. An artinian local ring (B, m) with $m^2 = 0$ is of this type.

In the theorem we are going to prove here, again the homomorphisms $I/I^2 \to B$ play an essential role. This is not astonishing since the definition of strict includes

Received by the editors October 16, 1980 and, in revised form, April 9, 1981.
1980 *Mathematics Subject Classification*. Primary 13H99; Secondary 13D10.

properties of algebras specializing to B. Such algebras however are nonobstructed deformations of B. The tangent space of the deformation functor of B is given by $T^2(B/k, B)$, which is $(I/I^2)^* = \mathrm{Hom}_B(I/I^2, B)$ modulo homomorphisms, induced by derivations.

To be more precise, the partial derivatives give rise to homomorphisms

$$\partial_i\colon I/I^2 \to B, \qquad f \bmod I^2 \to (\partial f/\partial X_i) \bmod I.$$

Letting U be the submodule of $(I/I^2)^*$ generated by the ∂_i, the $T^1(B/k, B) = (I/I^2)^*/U$. For more details see [4, 8].

Observe that if I is a homogeneous ideal, then $(I/I^2)^*$ has a natural grading: $\phi \in (I/I^2)^*$ is called homogeneous if, for all homogeneous elements $x \in I/I^2$, $\phi(x)$ is again homogeneous and $\deg \phi(x) - \deg x$ is independent of x. In that case one defines $\deg \phi = \deg \phi(x) - \deg x$.

In particular we have $\deg \partial_i = -1$ for $i = 1, \ldots, n$. Therefore also $T^1(B/k, B)$ has a natural grading if B is a graded algebra. If M is any graded B-module, then M_ν denotes its νth homogeneous part and $H_M(z) = \sum_{\nu \in \mathbf{Z}} \dim_k M_\nu z^\nu$ the Hilbert function of M.

Note that $(I/I^2)^*_\nu = T(B/k, B)_\nu$ for $\nu < -1$.

We are now able to state the main result of the paper.

THEOREM 1. *Let B be a graded artinian algebra such that $T^1(B/k, B)_\nu = 0$ for $\nu < -1$. Then B is strict.*

For the proof we will use a result of [3]. In our special case it can be described as follows: Let $B = k[|X_1 \ldots X_n|]/I$ be a graded artinian algebra with $I = (f_1, \ldots, f_k)$ and all f_i homogeneous. Suppose A specializes to B, i.e. $B \simeq A/(\underline{t})A$ where $\underline{t} = t_1, \ldots, t_m$ is a regular A-sequence. Then A can be written as

$$A \simeq k[|X_1, \ldots, X_n, T_1, \ldots, T_m|]/J$$

where $t_i = T_i \bmod J$ for $i = 1, \ldots, m$. The sequence $\underline{t}$ is called *super-regular* if the sequence $\underline{t}^* = t_1^*, \ldots, t_m^*$ of initial forms is again regular. In that case one has of course that $\mathrm{gr}_n(A)$ is Cohen-Macaulay and $\mathrm{gr}_n(A)/(\underline{t}^*)\mathrm{gr}_n(A) \simeq \mathrm{gr}_m(B)$.

THEOREM 2 [2, Theorem 1]. *$\underline{t}$ is super-regular if and only if there exists $F_i \in J$ such that, for $i = 1, \ldots, k$, $f_i = F_i \bmod (T_1, \ldots, T_m)$ and $\deg f_i = \deg F_i$.*

PROOF OF THEOREM 1. Let $B = k[|X_1, \ldots, X_n|]/I$ with $I = (f_1, \ldots, f_k)$ and all f_i homogeneous. Suppose A specializes to B. As in the discussions above we write $A \simeq k[|X_1 \ldots X_n, T_1, \ldots, T_m|]/J$ with $J = (F_1, \ldots, F_k)$ and we suppose that $\underline{t} = t_1, \ldots, t_m$ is a regular sequence with $B \simeq A/(\underline{t})A$, where $t_i = T_i \bmod I$.

We have

$$F_i = \sum_\nu f_i^{(\nu)} T^\nu, \qquad f_i^{(\nu)} \in k[|X_1, \ldots, X_n|]$$

and $f_i^{(0)} = f_i$ for $i = 1, \ldots, k$. Here ν denotes a multi-index.

Let $\overline{F_i} = \sum_\nu \bar{f}_i^{(\nu)} T^\nu$ $(1 \leqslant i \leqslant k)$, where $\bar{g}$ is the I-residue of an element $g \in k[|X_1, \ldots, X_n|]$. We are going to show that $\deg \overline{F_i} \geqslant \deg f_i$ for $i = 1, \ldots, k$. Then, of course, we can modify the F_i such that $\deg F_i = \deg f_i$ for $i = 1, \ldots, k$. Applying Theorem 2 we see that B is strict.

We will use the following notation: If $F \in k[|X_1, \ldots, X_n, T_1, \ldots, T_m|]$ with $F = \sum_\nu f^{(\nu)}T^\nu, f^{(\nu)} \in k[|X_1, \ldots, X_n|]$, then for $q \geqslant 0$

$$F^{(q)} = \sum_{|\nu| < q} f^{(\nu)}T^\nu$$

denotes the truncated series, where $|\nu| \leqslant q$ means that $\sum_{i=1}^n \nu_i \leqslant q$ for $\nu = (\nu_1, \ldots, \nu_n)$.

Assume now that $\deg \overline{F_i} < \deg f_i$ for some $i = 1, \ldots, k$. After renumbering we may even assume that for some $p \geqslant 0$ we have

$$(*) \qquad \deg \overline{F_i}^{(p)} \geqslant \deg f_i \quad \text{for } i = 1, \ldots, k \quad \text{and} \deg \overline{F_1}^{(p+1)} < \deg f_1.$$

Since k is infinite we may choose an automorphism

$$\Phi\colon k[|X_1, \ldots, X_n, T_1, \ldots, T_m|] \to k[|X_1, \ldots, X_m, T_1, \ldots, T_m|]$$

which is the identity on $k[|X_1, \ldots, X_n|]$ and such that for $\Phi(\overline{F_i}^{(p+1)}) = \sum_\nu \bar{g}^{(\nu)}T^\nu$ we have $\deg \bar{g}^{(p+1,0,\ldots,0)}T_1^{p+1} < \deg f_i$. We could take for instance the automorphism Φ defined by $\Phi(T_1) = \alpha_1 T_1$, $\Phi(T_i) = \alpha_i T_1 + T_i$ $(1 \leqslant i \leqslant m)$ with $\alpha_i \in k \setminus \{0\}$ suitably chosen.

After the application of such an automorphism Φ and after reduction modulo the new variables $\Phi(T_2), \ldots, \Phi(T_n)$, we may assume that we are dealing only with one variable T. Hence $A = k[|X_1, \ldots, X_n, T|]/(F_1, \ldots, F_k)$, $t = T \bmod I$ is a regular element of A and $B = A/tA$, the assumptions on the F_i being as before in $(*)$.

Next we may modify the F_i such that all the homogeneous components of the $f_i^{(\nu)}$ belonging to I are zero for $\nu \neq 0$.

We then obtain

$$(**) \qquad \deg F_i^{(p)} = \deg f_i \quad \text{for } i = 1, \ldots, k \quad \text{and} \quad \deg F_1^{(p+1)} < \deg f_1.$$

Since t is a regular element of A, the homomorphisms

$$\phi_q\colon k[T]/T^q \to A/t^qA, \qquad T \bmod T^q \to t \bmod t^q$$

are flat for all q. It is clear that $\phi_1\colon k \to A/tA = B$ is just the structure map of B and that $\phi_{q+1} \otimes k[T]/T^q = \phi_q$ for all q.

Consequently each relation $(r_i) = (r_1, \ldots, r_n)$ of $(f_1, \ldots, f_n)$ admits a lifting $(R_i^{(q)})$ of order q. This means that there exist polynomials

$$R_i^{(q)} = r_i^{(0)} + r_i^{(1)}T + \cdots + r_i^{(q)}T^q$$

with $r_i^{(\nu)} \in k[|X_1, \ldots, X_n|]$, $r_i^{(0)} = r_i$ and $\sum_{i=1}^n R_i^{(q)}F_i^{(q)} = 0 \bmod T^{q+1}$.

Moreover each lifting $(R_i^{(q)})$ of order q can be lifted further to $(R_i^{(q+1)})$, where $R_i^{(q+1)} = R_i^{(q)} + r_i^{(q+1)}T^{(q+1)}$, $r_i^{(q+1)} \in k[|X_1, \ldots, X_n|]$ and $\sum_{i=1}^k R_i^{(q+1)}F_i^{(q+1)} = 0 \bmod T^{q+2}$.

These are easy facts on flat modules and can be found for instance in [1, §3]. Now let (r_i) be a homogeneous relation of $(f_1, \ldots, f_n)$. Then $\deg r_i + \deg f_i$ does not depend on i. We put $d = \deg r_i + \deg f_i$.

We claim that for p as in $(**)$ we can find a lifting $(R_i^{(p)})$ of (r_i) of order p such that $\deg R_i^{(p)} = \deg r_i$ for $i = 1, \ldots, k$. In fact, let $(\overline{R}_i^{(p)})$ be any lifting of (r_i) of

order p. We proceed by induction and may assume that we have already

$$\deg \bar{R}_i^{(p-1)} = \deg r_i.$$

Since the coefficient of T^p in $\Sigma_{i=1}^p \bar{R}^{(p)} F_i^{(p)}$ must be zero, we obtain an equation

$$\sum_{i=1}^{k} \left(\sum_{\nu=0}^{p-1} \bar{r}_i^{(\nu)} f_i^{(p-\nu)} \right) + \sum_{i=1}^{k} \bar{r}_i^{(p)} f_i = 0.$$

Denoting by $(F)_l$ the lth homogeneous component of a series we find that

$$\left(\sum_{i=1}^{k} \bar{r}_i^{(p)} f_i \right)_l = 0 \quad \text{for } l < d - p.$$

We put $r_i^{(p)} = \Sigma_{l \geqslant \deg r_i - p} (\bar{r}_i^{(p)})_l$ and $R_i^{(p)} = \bar{R}_i^{(p-1)} + r_i^{(p)} T^p$. Then $(R_i^{(p)})$ is the relation we wanted. Next let $(R_i^{(p+1)})$ be a lifting of $(R_i^{(p)})$, where $\deg R_i^{(p)} = \deg r_i$ and $R_i^{(p+1)} = R_i^{(p)} + r_i^{(p+1)} T^{p+1}$ for $i = 1, \ldots, k$.

Again we have

$$\sum_{i=1}^{k} \left(\sum_{\nu=1}^{p} r_i^{(\nu)} f_i^{(p+1-\nu)} \right) + \sum_{i=1}^{k} r_i^{(p+1)} f_i + \sum_{i=1}^{k} r_i f_i^{(p+1)} = 0.$$

We also have

$$\deg \sum_{i=1}^{k} \left(\sum_{\nu=1}^{p} r_i^{(\nu)} f_i^{(p+1-\nu)} \right) \geqslant d - (p+1)$$

and by assumption (**) $\deg f_1^{(p+1)} < \deg f_1 - (p+1)$. Let $\lambda = \deg r_1 + \deg f_1^{(p+1)}$. Then $\lambda < d - (p+1)$ and

$$\left(\sum_{i=1}^{k} r_i^{(p+1)} f_i + \sum_{i=1}^{k} r_i f_i^{(p+1)} \right)_\lambda = 0.$$

Letting $\lambda_i = \lambda - \deg r_i$ and $g_i = (f_i^{(p+1)})_{\lambda_i}$, then $g_1 \neq 0$ and hence $g_1 \notin I$ but $\Sigma_{i=1}^k r_i g_i \in I$. The λ_i do not depend on the particular chosen relation (r_i). Thus

$$\phi\colon I/I^2 \to B, \qquad f_i \bmod I^2 \mapsto g_i \bmod I$$

is a homomorphism $\neq 0$ with $\deg \phi = \deg g_1 - \deg f_1 < -(p+1) \leqslant -1$. This contradicts our assumption that $T^1(B/k, B)_\nu = 0$ for $\nu < -1$.

The referee of this paper indicated to the author a proof of this theorem in which he does not use any reduction to the case of one variable T. The advantage of his proof is that no assumptions on k are necessary.

The following result, which provides a large class of strict algebras, was obtained by the author together with D. Eisenbud.

PROPOSITION 1. *Let $B = k[|X_1, \ldots, X_n|]/(f_1, \ldots, f_k)$ be an artinian algebra with $\deg f_i = d$ for $i = 1, \ldots, k$, where $n, d \geqslant 2$. If the module of relations is generated by homogeneous relations (r_i) with $\deg r_i = 1$ for $i = 1, \ldots, k$, then $T^1(B/k, B)_\nu = 0$ for $\nu < -1$.*

PROOF. We put $P = k[|X_1, \ldots, X_n|]$. $I = (f_1, \ldots, f_d)$ admits a homogeneous presentation $P^l(-d-1) \to P^k(-d) \to I \to 0$. Here $M(d)$ denotes as usual the shift of the graded module M by d, i.e. $M(d)_n = M_{d+n}$. Dualizing with respect to P and

with respect to $B = P/I$ we obtain for all integers ν a commutative diagram:

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}(I, P)_\nu & \to & P^k_{d+\nu} & \to & P^1_{d+1+\nu} \\ & & \alpha_0\downarrow & & \alpha_1\downarrow & & \alpha_2\downarrow \\ 0 & \to & \mathrm{Hom}(I, B)_\nu & \to & B^k_{d+\nu} & \to & B^1_{d+1+\nu} \end{array}$$

It follows that α_1, α_2 are isomorphisms for $\nu < -1$, hence also α_0.

Thus for $\nu < -1$ we get

$$\mathrm{Hom}_B(I/I^2, B)_\nu = \mathrm{Hom}_P(I, B)_\nu = \mathrm{Hom}_P(I, P)_\nu = \mathrm{Hom}_P(I, I)_\nu = 0.$$

Here we also used that grade $I \geqslant 2$.

We give two examples:

(a) Let $B = k[|X_1, \ldots, X_n|]/(X_1, \ldots, X_n)^d$ with $n \geqslant 2$. It is well known and easy to check that $(X_1, \ldots, X_n)^d$ is generated by the minors of the $d \times (d + n)$-matrix

$$\begin{pmatrix} X_1, & X_2, & \ldots, & X_n, & 0, & \ldots & 0 \\ 0, & X_1 & & \ldots, & X_n & & \\ \vdots & & & & & & \vdots \\ & & \ddots & & & \ddots & \\ 0, & \ldots & 0, & X_1 & & \ldots & X_n \end{pmatrix}$$

and that the generating relations of these minors are just the rows of this matrix. Hence by Proposition 1 and Theorem 1, B is strict.

(b) Let $B = k[|X_1, \ldots, X_n|]/(f_1, \ldots, f_k)$ with $n \geqslant 3$ and $\deg f_i = d > 1$ for $i = 1, \ldots, k$. Assume further that B is an artinian Gorenstein ring, that σ generates the socle of B and that $\deg \sigma = 2d - 2$. In [7] P. Schenzel calls Gorenstein rings of this type extremal. In fact they are extremal in the sense that in general one only has the inequality $\deg \sigma \geqslant 2d - 2$. P. Schenzel points out that extremal Gorenstein rings with given d occur as rings associated to certain polytops. For details see also [9]. P. Schenzel also proves a structure theorem (Theorem B) on the resolution of extremal Gorenstein rings. For the convenience of the reader we prove here what we need. Let

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0$$

be a minimal homogeneous P-resolution of B, $P = k[|X_1, \ldots, X_n|]$ as before. Since B is a Gorenstein ring the resolution is symmetric and in particular $F_n = P(-d_n)$ is a free module of rank 1. Say $F_i = \bigoplus_{j=1}^{\beta_i} P[-d_{ij}]$; then $d_{1j} = d$ for $j = 1, \ldots, \beta_1$ and again by symmetry $d_{n-1,j} = d_n - d$ for $j = 1, \ldots, \beta_{n-1}$.

Since the resolution is minimal we have the inequalities $d_{ij} \geqslant d + i - 1$ for $2 \leqslant i \leqslant n - 1$. It follows that $d_n \geqslant 2d + (n - 2)$. Calculating Hilbert functions it is easy to see that $\deg \sigma = d_n - n$; hence $\deg \sigma \geqslant 2d - 2$. Equality holds if and only if $d_{ij} = d + i - 1$ for $2 \leqslant 0 \leqslant n - 1$. In particular the module of relations of I is generated by linear relations.

The two above examples turn out to be special cases of a theorem due to D. Eisenbud and A. Iarrobino, which was communicated to the author by D. Eisenbud.

PROPOSITION 2. *Let $A = k[|X_1, \ldots, X_n|]$, H the Hilbert function of A, $B = A/I$, I generated by forms of degree $\geqslant d$. Suppose that the socle of A has μ_i generators of degree i for each i, and that $\sum_i \mu_i H(i-d) \leqslant H(d)$. Then equality holds, all the generators of I are of degree $d+1$ and the resolution is linear (except in the first and last term).*

We now reformulate slightly the condition $T^1(B/k, B)_\nu = 0$ for $\nu < -1$ in terms of the canonical module K_B of B. Again we assume that $B = k[|X_1, \ldots, X_n|]/I$ is an artinian graded algebra. Let $\sigma \in K_B$ generate the 1-dimensional socle of K_B. K_B is in a natural way a graded B-module with $\deg \sigma = 0$ and the canonical isomorphism

$$\operatorname{Hom}_B(I/I^2, B) \simeq \operatorname{Hom}_B(I \otimes_P K_B, K_B)$$

is an isomorphism of graded B-modules. Hence for the Hilbert functions we have the equality $H_{(I/I^2)^*}(z) = H_{I \otimes K_B}(z^{-1})$. It follows

PROPOSITION 3. *$T^1(B/k, B)_\nu = 0$ for $\nu < -1$ if and only if*

$$(I \otimes K_B)_\nu = 0 \quad \text{for } \nu > 1.$$

COROLLARY 1. *let B be a Gorenstein ring. Then $T^1(B/k, B)_\nu = 0$ for $\nu < -1$ if and only if $I^2 \subseteq (X_1, \ldots, X_n)^{c+2}$, where c is the degree of the socle of B.*

PROOF. In the Gorenstein case $K_B = B(c)$. Thus $T^1(B/k, B)_\nu = 0$ for $\nu < -1$ if and only if $(I/I^2)_{c+\nu} = 0$ for $\nu > 1$. This is equivalent to saying that $I_{c+\nu} = I^2_{c+\nu}$ for $\nu > 1$. Since $I_{c+\nu} = (m_B)_{c+\nu}$ for $\nu \geqslant 1$, the assertion follows.

COROLLARY 2. *Let B be an extremal Gorenstein ring (as in Example* (b)); *then*

$$I^2 = (X_1, \ldots, X_n)^{2g},$$

where g is the degree of the forms generating I.

PROOF. Obviously $I^2 \subseteq (X_1, \ldots, X_n)^{2g}$. Since $2g = c + 2$, the other inclusion follows from Corollary 1.

In order to state the following result we have to recall the notion of linkage studied by C. Peskine and L. Szpiro in [5]. The technique of linkage is a powerful tool to produce examples for whatever. Here we discuss only a special case of linkage.

Let $B_i = k[|X_1, \ldots, X_n|]/I_i$, $i = 1, 2$ be two artinian graded algebras. B_2 is said to be linked to B_1 with respect to a regular sequence $f = f_1, \ldots, f_n$ of forms if $f \subseteq I_1 \cap I_2$ and $I_2 = (f)\colon I_1$. In this case one also has $I_1 = (f)\colon I_2$. Thus linkage is a symmetric relation.

PROPOSITION 4. *Let $B_1 = k[|X_1, \ldots, X_n|]/I_1$ and $B_2 = k[|X_1, \ldots, X_n|]/I_2$ be linked with respect to a regular sequence f of forms of degree d. Then the following holds:*

(a)

$$\begin{aligned}\dim_k T^1(B_1/k, B_1)_\nu &- \dim_k T^1(B_2/k, B_2)_\nu \\ &= \dim_k (B_1)_{n(d-1)-d-\nu} - \dim_k (B_2)_{n(d-1)-d-\nu} \quad \text{for } \nu < -1.\end{aligned}$$

(b) $\dim_k T^1(B_1/k, B_1)_\nu = \dim_k T^1(B_2/k, B_2)_\nu$ *for* $\nu < -1$, *if and only if* $\dim_k(B_1)_\nu + \dim_k(B_1)_{n(d-1)-\nu} = \dim_k(k[|X_1, \ldots, X_n|]/(f))_\nu$ *for* $\nu \geqslant (n-1)d - n + 2$.

(c) *If* $(B_1)_\nu = 0$ *for* $\nu \geqslant (n-1)d - n + 2$ *and* $(B_1)_\nu = k[|X_1, \ldots, X_n|]_\nu$ *for* $\nu \leqslant d - 2$, *then*

$$\dim_k T^1(B_1/k, B_1)_\nu = \dim_k T^1(B_2/k, B_2)_\nu$$

for $\nu < -1$.

Of course one could also consider the case where not all the f_i in the linkage have the same degree. The proof for this more general case is the same, but obviously the resulting formulas will be even more technical.

Before proving the proposition we consider an explicit example. Let $I_1 = (X_1, \ldots, X_n)$ and $f = X_1^2, \ldots, X_n^2$. Then $B_1 = k$ and $d = 2$. Trivially the conditions (c) are satisfied. Furthermore it is easy to see that (f): $I_1 = (X_1^2, \ldots, X_n^2, X_1 \cdot \ldots \cdot X_n)$. Thus $B_2 = k[|X_1, \ldots, X_n|]/(X_1^2, \ldots, X_n^2, X_1 \cdot \ldots \cdot X_n)$ is strict, since $B_1 = k$ is strict. The example shows that for a strict algebra not necessarily all generators of the defining ideal must be of the same degree.

Proof of Proposition 4. The proof uses essentially an idea due to R. O. Buchweitz communicated to the author (see [2]). We write $B_1 = P/I_1$, $B_2 = P/I_2$ and $B = P/(f)$. Since $K_{B_2}(-n(d-1)) \simeq I_1/(f)$ we obtain the following exact sequence of graded P-modules:

$$0 \to K_{B_2}(-n(d-1)) \to B \to B_1 \to 0.$$

Tensorizing this sequence with B_2 over P we obtain the long exact sequence

$$\begin{aligned} \to \quad \mathrm{Tor}_2(B, B_2) \xrightarrow{\alpha_2} \mathrm{Tor}_2(B_1, B_2) \to \mathrm{Tor}_1(K_{B_2}(-n(d-1)), B_2) \\ \to \mathrm{Tor}_1(B, B_2) \xrightarrow{\alpha_1} \mathrm{Tor}_1(B_1, B_2) \to \end{aligned}$$

Exchanging the role of B_1 and B_2 we obtain a similar exact sequence

$$\begin{aligned} \mathrm{Tor}_2(B, B_1) \xrightarrow{\beta_2} \mathrm{Tor}_2(B_2, B_1) \to \mathrm{Tor}_1(K_{B_1}(-n(d-1)), B_1) \\ \to \mathrm{Tor}_1(B, B_1) \xrightarrow{\beta_1} \mathrm{Tor}_1(B_2, B_1) \to \end{aligned}$$

The crucial point is now that $\mathrm{Im}\,\alpha_i = \mathrm{Im}\,\beta_i$ for $i = 1, 2, \ldots$. In fact, consider the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Tor}_i(B, B_1) \arrow[r, "\beta_i"] & \mathrm{Tor}_i(B_2, B_1) \arrow[d, "\wr"] \\ \mathrm{Tor}_i(B, B) \arrow[ur, "\sigma_i"] \arrow[dr, "\tau_i"'] & & \\ & \mathrm{Tor}_i(B, B_2) \arrow[r, "\alpha_i"'] & \mathrm{Tor}_i(B_1, B_2) \end{tikzcd}$$

Using the Koszul complex to compute $\mathrm{Tor}_i(B, B)$, $\mathrm{Tor}_i(B, B_1)$ and $\mathrm{Tor}_i(B, B_2)$ one sees immediately that σ_i and τ_i are surjective. Hence the conclusion.

The above sequences now yield

$$\begin{aligned} &\dim_k \mathrm{Tor}_1\big(K_{B_2}(-n(d-1)), B_2\big)_\nu - \dim_k \mathrm{Tor}_1(B, B_2)_\nu \\ &\qquad = \dim_k \mathrm{Tor}_1\big(K_{B_1}(-n(d-1)), B_1\big)_\nu - \dim_k \mathrm{Tor}_1(B, B_1)_\nu. \end{aligned}$$

Observing that

$$\mathrm{Tor}_1(K_{B_i}(-n(d-1)), B_i) \simeq I_i \otimes K_{B_i}(-n(d-1)),$$

$$\mathrm{Tor}_1(B, B_i) \simeq B_i(-d) \quad \text{and} \quad H_{(I/I^2)^\bullet}(z) = H_{I \otimes K_B}(z^{-1}),$$

the assertion (a) follows easily.

(b) follows from (a) taking into account that the Hilbert function of B_2 can be derived from the Hilbert function of B_1. In fact, from the exact sequence $0 \to K_{B_2}(-n(d-1)) \to B \to B_1 \to 0$ we obtain

$$H_{B_1}(z) + z^{-n(d-1)} H_{K_{B_2}}(z) = H_B(z).$$

On the other hand $H_{K_{B_2}}(z) = H_{B_2}(z^{-1})$. Finally (c) is a simple consequence of (b).

It would be interesting to describe all graded artinian algebras B for which $T^1(B/k, B)_\nu = 0$ for $\nu < -1$. If for instance edim $B = 2$, then it is easy to see that $T^1(B/k, B)_\nu = 0$ for $\nu < -1$ if and only if all defining relations are of the same degree and B is not a complete intersection.

It seems to be even more complicated to describe all strict algebras. Theorem 1 gives only a sufficient condition for an algebra to be strict. The easiest example is $k[\varepsilon] = k[X]/(X^2)$. It is obviously strict; however $T^1(k[\varepsilon]/k, k[\varepsilon])_{-2} \neq 0$.

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