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CONSTRUCTION OF FIXED POINTS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

GREGORY B. PASSTY

ABSTRACT. In uniformly convex Banach spaces with Fréchet differentiable norms (e.g. L^p, $1 < p < \infty$), fixed points for asymptotically nonexpansive mappings are constructed as weak limits of iterates of the mappings themselves or of related mappings.

Let E be a uniformly convex Banach space, and let C be a closed convex subset of E. A mapping U of C into itself is said to be *asymptotically nonexpansive* (Goebel and Kirk [6]) if $\|U^n x - U^n y\| \leq k_n \|x - y\|$ for all x and y in C, with $\lim_n k_n = 1$. It was proved in [6] that if C is further assumed to be bounded, then an asymptotically nonexpansive self-map of C has a fixed point. We show here that if E has a Fréchet differentiable norm, and if U is, for example, weakly continuous, then fixed points of U can be obtained by iterating U starting at a point of asymptotic regularity.

Theorem 1 extends theorems of Feathers and Dotson [5] and of Bose [2] which were obtained in uniformly convex spaces with weakly continuous duality maps. The basic tool in both of these papers was Opial's Lemma [7]. Because this lemma does not carry over to L^p, $p \neq 2$, new techniques are needed for this more general case. These were provided by Baillon [1] and simplified by Bruck [4] when the norm is Fréchet differentiable.

We will present Theorem 1 in a slightly more general form, and then discuss applications to asymptotically nonexpansive mappings and to a conjecture of H. Schaefer [8].

First we extend the definition of [6] to sequences of maps which are not necessarily powers of a given map.

DEFINITION 1. The sequence $\{T_n\}_{n=1}^{\infty}$ of self-maps of C is asymptotically nonexpansive if $\|T_n x - T_n y\| \leq k_n \|x - y\|$ for all x, y in C with $\lim_n k_n = 1$.

Denote the set of fixed points of T by $F(T)$, strong convergence by $\to$, and weak convergence by $\rightharpoonup$. We may now state

THEOREM 1. *Let E be uniformly convex with a Fréchet differentiable norm, and C a closed convex subset of E. Let F be a subset of C and $S = \{T_n\}_{n=1}^{\infty}$ an asymptotically nonexpansive sequence of self-maps of C such that* (a) $F \subset \bigcap_{n=1}^{\infty} F(T_n)$. *Assume also that there exists x_0 in C for which*

(b) $T_{n_i} x_0 \underset{i}{\rightharpoonup} z$ *implies $z \in F$, and*

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(c) $T_n T_m x_0 - T_n x_0 \to 0$ *as* $n \to \infty$ *for all (fixed)* m.
Then either (i) $F = \varnothing$ *and* $\|T_n x_0\| \to +\infty$ *or* (ii) $F \neq \varnothing$ *and* $T_n x_0 \rightharpoonup$ *an element of* F.

Note that hypothesis (c) may be interpreted as asymptotic regularity of S at x_0. In order to prove Theorem 1, we proceed using several lemmas.

LEMMA 1. *Let* x_0 *and* y_0 *be any two elements of* C *for which hypothesis* (c) *holds. Then* $\lim_n \|T_n x_0 - T_n y_0\|$ *exists.*

PROOF. By the triangle inequality and the Lipschitz property of T_n,

$$\begin{aligned}\|T_n x_0 - T_n y_0\| &\leqslant \|T_n x_0 - T_n T_m x_0\| + \|T_n T_m x_0 - T_n T_m y_0\| + \|T_n T_m y_0 - T_n y_0\| \\ &\leqslant \|T_n x_0 - T_n T_m x_0\| + k_n \|T_m x_0 - T_m y_0\| + \|T_n T_m y_0 - T_n y_0\|.\end{aligned}$$

Fixing m and letting $n \to \infty$, applying (c), and then letting $m \to \infty$, we see

$$\limsup_n \|T_n x_0 - T_n y_0\| \leqslant \liminf_m \|T_m x_0 - T_m y_0\|. \quad \text{Q.E.D.}$$

COROLLARY 1. *For each* $f \in F$, $\lim_n \|T_n x_0 - f\|$ *exists.*

PROOF. By (a), $T_n f = f$ for all n. In addition, x_0 (by assumption) and f satisfy (c). Q.E.D.

A very important tool is a result proved by Bruck, which we state here as

LEMMA 2 [4]. *Let* E *be a uniformly convex Banach space and let* K *be a nonempty closed bounded convex subset of* E. *Then there exists a strictly increasing, continuous convex function* $\gamma\colon \mathbf{R}^+ \to \mathbf{R}^+$ *with* $\gamma(0) = 0$ *such that every nonexpansive mapping* $U\colon K \to E$ *satisfies*

$$\gamma(\|rUx + (1-r)Uy - U(rx + (1-r)y)\|) \leqslant \|x - y\| - \|Ux - Uy\|$$

for all x, y *in* K *and* $0 \leqslant r \leqslant 1$.

More suitable to our purposes is a variant of Lemma 2 for mappings which are not necessarily nonexpansive.

COROLLARY 2. *Let* E, K, *and consequently* γ *be as in Lemma* 2, *and let* $T\colon K \to E$ *be Lipschitz with Lipschitz constant* k. *Then*

$$\|rTx + (1-r)Ty - T(rx + (1-r)y)\| \leqslant k\gamma^{-1}(\|x - y\| - k^{-1}\|Tx - Ty\|)$$

for all x *and* y *in* K *and* $0 \leqslant r \leqslant 1$.

We now begin the proof of Theorem 1, following lines developed in [4].

PROOF OF THEOREM 1 (BEGINNING). Suppose some subsequence $\{T_{n_i} x_0\}$ is bounded. Since E is reflexive, a further subsequence must converge weakly to an element $z \in E$ which, by (b), is in F. Thus $F = \varnothing$ implies $\|T_n x_0\| \to \infty$.

If, on the other hand, $F \neq \varnothing$, then there is some $f_0 \in F$ and, by Corollary 1, $\{\|T_n x_0 - f_0\|\}$ is bounded, say, by R. Let $C_1 = \{x \in C\colon \|x - f_0\| \leqslant R\}$. Then C_1 is closed, convex, bounded, and nonempty. Furthermore, $T_n x_0 \in C_1$ for all n. We take C_1 to be the set K in Corollary 2.

We abstract the next section of the proof as

LEMMA 3. *Under the hypotheses of Theorem* 1, $\lim_n \|rT_nx_0 + (1-r)f_1 - f_2\|$ *exists for all* $f_1, f_2 \in F$ *and for all* $0 \leqslant r \leqslant 1$.

PROOF.

$$\begin{aligned}
\|rT_nx_0 + (1-r)f_1 - f_2\| &\leqslant r\|T_nx_0 - T_nT_mx_0\| + \|rT_nT_mx_0 + (1-r)f_1 - f_2\| \\
&\leqslant r\|T_nx_0 - T_nT_mx_0\| + \|rT_nT_mx_0 + (1-r)T_nf_1 - T_n(rT_mx_0 + (1-r)f_1)\| \\
&\quad + \|T_n(rT_mx_0 + (1-r)f_1) - T_nf_2\| \\
&\leqslant r\|T_nx_0 - T_nT_mx_0\| + k_n\gamma^{-1}\big(\|T_mx_0 - f_1\| - k_n^{-1}\|T_nT_mx_0 - f_1\|\big) \\
&\quad + k_n\|rT_mx_0 + (1-r)f_1 - f_2\| \\
&\leqslant r\|T_nx_0 - T_nT_mx_0\| \\
&\quad + k_n\gamma^{-1}\big(\|T_mx_0 - f_1\| - k_n^{-1}\|T_nx_0 - f_1\| + k_n^{-1}\|T_nx_0 - T_nT_mx_0\|\big) \\
&\quad + k_n\|rT_mx_0 + (1-r)f_1 - f_2\|.
\end{aligned}$$

Here we have used the triangle inequality, hypothesis (a), the Lipschitz property of T_n, Corollary 2, the triangle inequality again, and the fact that γ^{-1} is also an increasing function. Fix m and let $n \to \infty$:

$$\begin{aligned}
\limsup_n \|rT_nx_0 + (1-r)f_1 - f_2\| \leqslant 0 + \gamma^{-1}\Big(\|T_mx_0 - f_1\| - \lim_n \|T_nx_0 - f_1\|\Big) \\
+ \|rT_mx_0 + (1-r)f_1 - f_2\|.
\end{aligned}$$

Now letting $m \to \infty$,

$$\limsup_n \|rT_nx_0 + (1-r)f_1 - f_2\| \leqslant \liminf_m \|rT_mx_0 + (1-r)f_1 - f_2\|. \quad \text{Q.E.D.}$$

PROOF OF THEOREM 1 (CONCLUSION). Let $f_1, f_2 \in F$. Since E has a Fréchet differentiable norm, we may take $J(u)$ to be the Fréchet derivative of $\frac{1}{2}\|\cdot\|^2$ at u. Then there exists an increasing function λ: $\mathbf{R}^+ \to \mathbf{R}^+$ such that $t^{-1}\lambda(t) \to 0$ as $t \to 0^+$ and

$$\begin{aligned}
\tfrac{1}{2}\|f_1 - f_2\|^2 + (J(f_1 - f_2), h) &\leqslant \tfrac{1}{2}\|f_1 - f_2 + h\|^2 \\
&\leqslant \tfrac{1}{2}\|f_1 - f_2\|^2 + (J(f_1 - f_2), h) + \lambda(\|h\|)
\end{aligned}$$

for all $h \in E$. Take $h = r(T_nx_0 - f_1)$. Then

$$\begin{aligned}
\tfrac{1}{2}\|f_1 - f_2\|^2 + r(J(f_1 - f_2), T_nx_0 - f_1) &\leqslant \tfrac{1}{2}\|rT_nx_0 + (1-r)f_1 - f_2\|^2 \\
&\leqslant \tfrac{1}{2}\|f_1 - f_2\|^2 + r(J(f_1 - f_2), T_nx_0 - f_1) + \lambda(r\|T_nx_0 - f_1\|).
\end{aligned}$$

Letting $n \to \infty$ and using Lemma 3, we obtain

$$\begin{aligned}
\limsup_n r(J(f_1 - f_2), T_nx_0 - f_1) &\leqslant \tfrac{1}{2}\lim_n \|rT_nx_0 + (1-r)f_1 - f_2\|^2 - \tfrac{1}{2}\|f_1 - f_2\|^2 \\
&\leqslant \liminf_n r(J(f_1 - f_2), T_nx_0 - f_1) + \lambda(rM),
\end{aligned}$$

where $\|T_n x_0 - f_1\| \leq M$ for all n. Dividing by r and letting $r \to 0^+$ shows $\lim_n(J(f_1 - f_2), T_n x_0)$ exists for all $f_1, f_2 \in F$. Thus if z_1 and z_2 are two weak subsequential limits of $\{T_n x_0\}$, $(J(f_1 - f_2), z_1 - z_2) = 0$. By (b), z_1 and z_2 are in F; thus we may take $f_i = z_i$ for $i = 1, 2$, finding that $0 = (J(z_1 - z_2), z_1 - z_2) = \|z_1 - z_2\|^2$. (We have used $(Ju, u) = \|u\|^2 = \|Ju\|^2$ for all $u \in E$ [3, p. 97].) Since all weak subsequential limits of the bounded sequence $\{T_n x_0\}$ are thus equal, $\{T_n x_0\}$ must converge weakly to an element of F. Q.E.D.

We now present a consequence of Theorem 1 for weakly continuous mappings.

COROLLARY 3. *Let C be closed, convex, and bounded, and let U: $C \to C$ be weakly continuous and asymptotically nonexpansive. If U is asymptotically regular at $x_0 \in C$ (i.e., if $U^{n+1}x_0 - U^n x_0 \to 0$), then $U^n x_0$ converges weakly to a fixed point of U.*

PROOF. In Theorem 1, we take $F = F(U)$, and $S = \{U^n\}_{n=1}^{\infty}$. It is obvious that (a) holds, and (c) holds by assumption. To see that (b) holds, note that $U^{n_j}x_0 \rightharpoonup z$ and weak continuity imply that $U^{n_j+1}x_0 \rightharpoonup Uz$. On the other hand, asymptotic regularity implies that $\{U^{n_j+1}x_0\}$ and $\{U^{n_j}x_0\}$ must have the same limit, namely z. Thus $Uz = z$. $F \neq \varnothing$ by [6], so $\{U^n x_0\}$ must converge weakly to an element of F. Q.E.D.

Theorem 1 may also be used to make statements about nonexpansive mappings.

THEOREM 2. *Let E be uniformly convex with a Fréchet differentiable norm, let C be a bounded closed convex subset of E, and let U be a nonexpansive self-map of C.*

Set $U_\lambda = \lambda U + (1 - \lambda)I$ for fixed $0 < \lambda < 1$. Then for any $x \in C$, $\{U_\lambda^n x\}$ converges weakly to a fixed point of U.

REMARK. This theorem shows that Schaefer's conjecture [8] holds in a class of uniformly convex spaces which includes L^p, $1 < p < \infty$.

PROOF OF THEOREM 2. We apply Theorem 1 with $F = F(U)$ and $S = \{U_\lambda^n\}_{n=1}^{\infty}$. $F(U) = F(U_\lambda)$ for all $0 < \lambda < 1$, so (a) holds. By the Browder-Kirk-Göhde Theorem, U has a fixed point in C. U_λ is thus asymptotically regular [8, Lemma 2] establishing (c) with x_0 any element of C. In order to verify (b), note that,by asymptotic regularity, $(I - U_\lambda)U_\lambda^{n_j}x \to 0$. Since $U_\lambda^{n_j}x \rightharpoonup z$ (by assumption) and since $(I - U_\lambda)$ is demiclosed [3, Theorem 8.4], we see that $0 = (I - U_\lambda)z$; i.e. $z = Uz$. Since F is nonempty, $\{U_\lambda^n x\}$ must converge weakly to a fixed point of U. Q.E.D.

ADDED IN PROOF. The author has learned that Theorem 2 is a special case of a result of S. Reich (J. Math. Anal. Appl. **67** (1979), 274–276, Theorem 2).

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SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GEORGIA 30332