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THE ASYMPTOTIC BEHAVIOR OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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ABSTRACT. Let $u'' + f(t,u) = 0$ be a nonlinear differential equation. If there are two nonnegative continuous functions $v(t)$, $\varphi(t)$ for $t \geqslant 0$, and a continuous function $g(u)$ for $u \geqslant 0$, such that (i) $\int_1^\infty v(t)\varphi(t)\,dt < \infty$; (ii) for $u > 0$, $g(u)$ is positive and nondecreasing; (iii) $|f(t,u)| < v(t)\varphi(t)g(|u|/t)$ for $t \geqslant 1$, $-\infty < u < \infty$, then the equation has solutions asymptotic to $a + bt$, where a, b are constants and $b \neq 0$. Our result generalizes a theorem of D. S. Cohen [3].

Consider the nonlinear differential equation

$$u'' + f(t, u) = 0. \tag{1}$$

D. S. Cohen [3] proved the following theorem.

THEOREM A. *Suppose $f(t, u)$ satisfies the following conditions:*
(H-1) *$f(t, u)$ is continuous on D: $t \geqslant 0$, $-\infty < u < \infty$.*
(H-2) *the derivative $f_u(t, u)$ exists on D and $f_u(t, u) > 0$ on D.*
(H-3) *$|f(t, u)| < f_u(t, 0)|u|$ on D.*
In addition, suppose that

$$\int_1^\infty t f_u(t, 0)\, dt < \infty. \tag{2}$$

Then equation (1) has solutions which are asymptotic to $a + bt$ as $t \to \infty$, where a, b are constants and $b \neq 0$.

In the proof of Theorem A, Cohen used R. Bellman's method [1, pp. 114–115] and Gronwall's inequality. In this paper we use the same method and Bihari's inequality [2] to generalize Theorem A.

THEOREM B. *Let $f(t, u)$ be continuous on D: $t \geqslant 0$, $-\infty < u < \infty$. If there are two nonnegative continuous functions $v(t)$, $\varphi(t)$ for $t \geqslant 0$, and a continuous function $g(u)$ for $u \geqslant 0$, such that*
(i) *$\int_1^\infty v(t)\varphi(t)\, dt < \infty$,*
(ii) *for $u > 0$, $g(u)$ is positive and nondecreasing,*
(iii) *$|f(t, u)| < v(t)\varphi(t)g(|u|/t)$ for $t \geqslant 1$, $-\infty < u < \infty$,*
then the equation (1) has solutions which are asymptotic to $a + bt$, where a, b are constants and $b \neq 0$.

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REMARK. If we let $v(t) = f_u(t, 0)$, $\varphi(t) = t$, $g(u) = u$ in Theorem B, we obtain Theorem A.

PROOF OF THEOREM B. Integrating (1) twice on $[1, t]$, we have

$$u(t) = c_1 + c_2 t - \int_1^t (t - s) f(s, u(s))\, ds. \tag{3}$$

Choose $c_1 > 1$ and let $c_3 = c_1 + |c_2|$. Then for $t > 1$ we have

$$\begin{aligned} \frac{|u(t)|}{t} &\leqslant c_3 + \int_1^t f(s, u(s))\, ds \\ &\leqslant c_3 + \int_1^t v(s)\varphi(s) g(|u(s)|/s)\, ds. \end{aligned}$$

By Bihari's inequality we have

$$\frac{|u(t)|}{t} \leqslant G^{-1}\left(G(c_3) + \int_1^t v(s)\varphi(s)\, ds\right). \tag{4}$$

Here $G(x) = \int_1^x dt/g(t)$, $G^{-1}(x)$ is the inverse function of $G(x)$. From $g(t) > 0$ we know that $G(x)$ is increasing; hence $G^{-1}(x)$ exists, and is also increasing.

Now let $c_4 = G(c_3) + \int_1^\infty v(s)\varphi(s)\, ds$. Since $G^{-1}(x)$ is increasing, we have

$$\frac{|u(t)|}{t} \leqslant G^{-1}(c_4). \tag{5}$$

Differentiating (3), we have

$$u'(t) = c_2 - \int_1^t f(s, u(s))\, ds. \tag{6}$$

By conditions (i), (ii), (iii) and (5), we have

$$\begin{aligned} \int_1^t |f(s, u(s))|\, ds &\leqslant \int_1^t v(t)\varphi(t) g(|u(s)|/s)\, ds \\ &\leqslant g\left(G^{-1}(c_4)\right) \int v(s)\varphi(s)\, ds < \infty. \end{aligned}$$

Therefore $u'(t) \to c_2 - \int_1^\infty f(s, u(s))\, ds$ as $t \to \infty$. If we choose c_2 sufficiently large, then $u'(t) > 1$. Hence $\lim_{t\to\infty} u'(t) \neq 0$. This proves the theorem.

We give an example to which Theorem B applies but Theorem A does not.

EXAMPLE.

$$u'' + t^{-4} u^2 \cos u = 0. \tag{7}$$

Since $f_u(t, 0) = 0$, (H-3) does not hold and Theorem A does not apply. Let $v(t) = t^{-4}$, $\varphi(t) = t^2$, $g(u) = u^2$. Then conditions (i), (ii) and (iii) are satisfied and equation (7) has solutions asymptotic to $a + bt$ as $t \to \infty$.

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