PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, February 1982

FINITELY GENERATED CODINGS AND THE DEGREES R.E. IN A DEGREE d

RICHARD A. SHORE[1]

ABSTRACT. We introduce finitely generated (partial) lattices which can be used to code an arbitrary set D. Results of Lerman, Shore and Soare are used to embed these lattices in the degrees r.e. in D. Thus if the degrees r.e. in and above **d** are isomorphic to those r.e. in and above **c**, **d** and **c** are of the same arithmetic degree. Similar applications are given to generic degrees and general homogeneity questions.

After the fact one might say that the main purpose of this paper is to introduce some schemes for doing coding in degree theory via finitely generated sets of degrees as opposed to the usual methods that employ definable substructures. Indeed we will describe such schemes and a number of applications to problems in degree theory. Truthfully, however, the motivation for this paper was the conjecture of Sacks [**1966**, p. 171] that RED(**a**), the degrees recursively enumerable in and above **a**, are isomorphic to RED = RED(**0**) for every degree **a**. Our main result will refute this conjecture. Indeed we will show that if RED(**a**) $\cong$ RED(**b**) then **a** and **b** are contained in the same arithmetic degree.

As we have said that Sacks [**1966**] supplies the question behind this paper we should also note that Lerman, Shore and Soare [**1981**] will essentially supply the answer. In that paper we proved that the r.e. degrees are not $\aleph_0$-categorical by embedding distinct partial lattices $\mathcal{P}_n$ in RED, all of which were generated (under $\wedge$ and $\vee$) by three elements. We will here use the natural limit $\mathcal{P}_\omega$ of these structures as the core of our coding scheme. Before describing $\mathcal{P}_\omega$ we restate the definition of a partial lattice.

DEFINITION 1. A structure $\mathcal{P} = \langle P, \leqslant, \nleqslant, \vee, \wedge \rangle$ is a *partial lattice* if there is a partial order on P containing $\leqslant$ and disjoint from $\nleqslant$ such that

(1.1) $\forall a, b, c \in P\ (a \vee b = c \rightarrow c$ is the least upper bound of a and b in P),

(1.2) $\forall a, b, c \in P\ (a \wedge b = c \rightarrow c$ is the greatest lower bound of a and b in P).

(Note that we are treating $\wedge$ and $\vee$ as partial functions.)

Now $\mathcal{P}_\omega$ has three generators t_0^0, t_1^0 and t_2^0. Its elements are t_0^n, t_1^n, t_2^n, b_0^n, b_1^n and b_2^n for $n \in N$. The defining relations determining the structure of $\mathcal{P}_\omega$ are as follows.

Received by the editors October 30, 1980 and, in revised form, March 10, 1981. Presented to the Society at the Special Session in Recursion Theory at the October 1980 meeting in Kenosha, Wisconsin.

1980 *Mathematics Subject Classification*. Primary 03D25, 03D30.

Key words and phrases. Recursively enumerable degrees, homogeneity problems, generic degrees.

[1]The preparation of this paper was partially supported by NSF grant MCS80-03016. We would also like to thank Carl Jockusch, Jr. for several helpful comments and suggestions.

(2.1) $t_i^n \wedge t_j^n = b_k^n$ for $n \geqslant 0$, $\{i,j,k\} = \{0,1,2\}$, and

(2.2) $t_i^n = t_i^{n-1} \vee b_i^{n-1}$ for $n \geqslant 1$, $i < 3$.

The ordering is the obvious one induced by these relations and the requirement that all the named elements be distinct. See Figure 1 for a picture of $\mathcal{P}_\omega$ in which solid lines indicate infimums and dashed ones supremums.

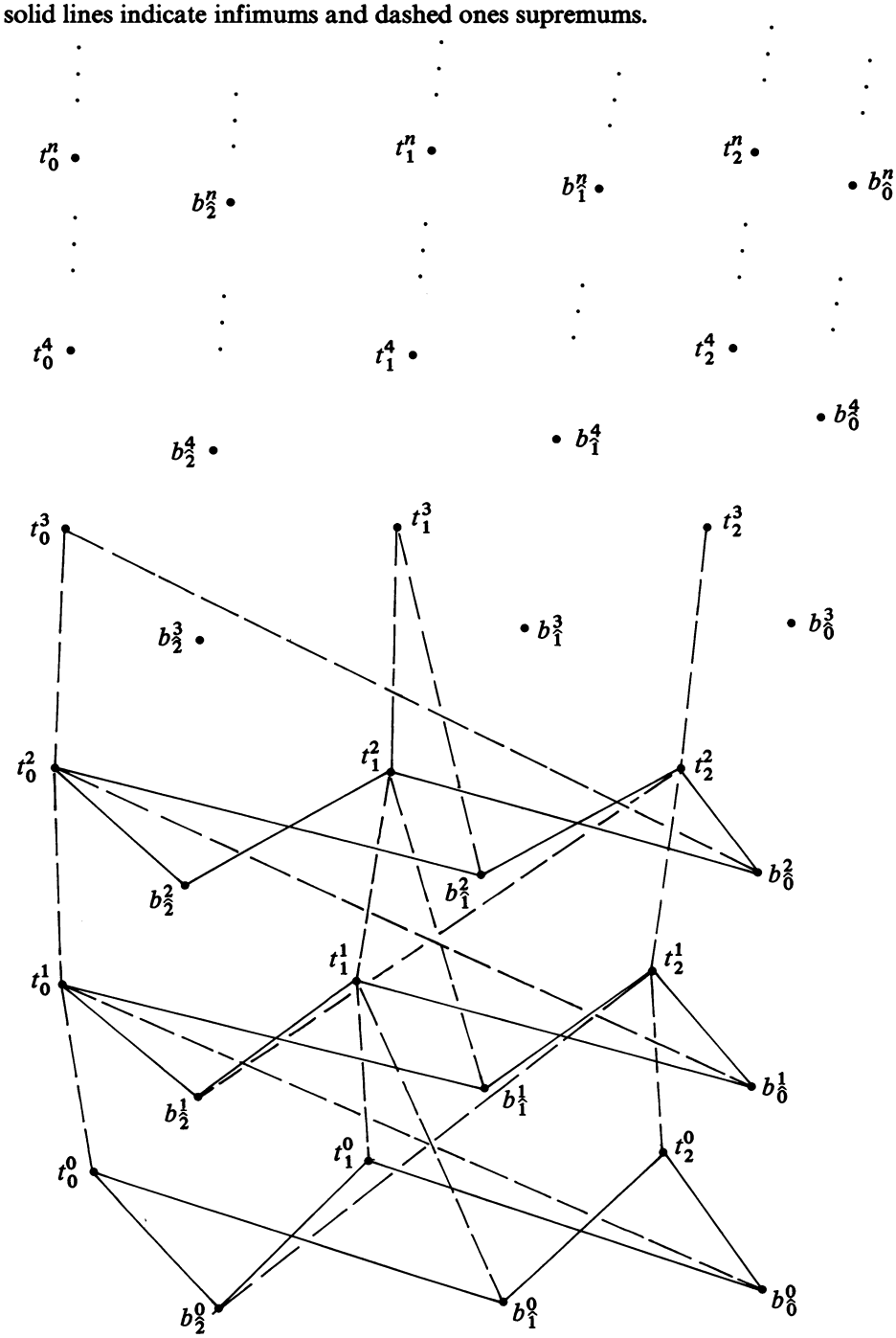

FIGURE 1

We can now code an arbitrary set A into a finitely generated partial lattice $\mathcal{P}_A$ by adding on one more generator a incomparable with all the old elements, other new but not necessarily distinct elements c_n, and new relations given by

(2.3) $c_n = a \vee t_2^n$,

(2.4) $n \in A \Leftrightarrow b_2^n \leqslant c_n$,

with all of the appropriate induced ordering relations and nonordering relations elsewhere. We claim that we can effectively recover the set A from the jump of any presentation of $\mathcal{P}_A$. Thus if $\mathcal{P}_A$ can be embedded in some upper semilattice $\mathcal{L}$, with $\leqslant$, $\nleqslant$, $\vee$ and $\wedge$ preserved, then A is arithmetic in any presentation of $\mathcal{L}$. The point here is that we can start off with the elements representing a, t_0^0, t_1^0 and t_2^0. Assuming we have inductively calculated t_0^n, t_1^n and t_2^n it only takes a search recursive in the jump of the presentation of $\mathcal{L}$ even as a partial ordering to find the b_i^n and c_n. We then ascertain if $n \in A$ by asking if $b_2^n \leqslant c_n$. To continue the induction we then find $t_i^{n+1} = t_i^n \vee b_i^n$ effectively in the presentation.

Now the proof of the main theorem of Lerman, Shore and Soare [**1981**] actually shows that any recursively presented partial lattice $\mathcal{P}$ having a property called TPP can be embedded in RED.

DEFINITION 3. A partial lattice $\mathcal{P}$ has the *trace probe property* (*TPP*) if, for all $p, q \in P$ such that $p \nleqslant q$, there is a finite sequence $Q_0, \{q_0\}, Q_1, \{q_1\}, \ldots, Q_n$ of subsets of P such that

(3.1) $Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_n$.

(3.2) $p \in Q_0$ and for all $r \in Q_0 \cup \{q_0\}$ it is not the case that $r \leqslant q$.

(3.3) For all $i < n$, $Q_i \cup \{q_i\}$ and $\{q_i\}$ are trace complete as is Q_n. ($Q \subseteq P$ is trace complete if, for all $q \in Q$ and $a, b, c \in P$ with $q \leqslant c = a \vee b$, there is a $p \in Q$ such that $p \leqslant a$ or $p \leqslant b$.)

(3.4) For all $a, b, c \in P$ such that $a \wedge b = c$ and $i \leqslant n$, if there are $a_1, b_1 \in Q_i$ such that $a_1 \leqslant a$ and $b_1 \leqslant b$, then either $i = 0$ and there is an $r \in Q_0$ such that $r \leqslant c$, or $i > 0$ and either there is an $r \in Q_{i-1}$ with $r \leqslant c$ or no element of $Q_{i-1} \cup \{q_{i-1}\}$ is $\leqslant a$ or no element of $Q_{i-1} \cup \{q_{i-1}\}$ is $\leqslant b$.

(3.5) For all $a, b, c \in P$ such that $a \wedge b = c$ and all $i < m$, either not $q_i \leqslant a$ or not $q_i \leqslant b$.

Relativizing the proof shows that if $\mathcal{P}$ is **d**-presentable and has TPP then it can be embedded in RED(**d**). As for the $\mathcal{P}_n$ it is a bit easier to use a modified version $\mathcal{P}_A^*$ of $\mathcal{P}_A$ when verifying TPP. $\mathcal{P}_A^*$ has the same elements as $\mathcal{P}_A$ and $\vee$, $\wedge$ and $\leqslant$ also are identical. We define $\nleqslant$ to hold in $\mathcal{P}_A^*$ only in the following cases:

(2.5) $b_j^m \nleqslant t_j^m$ for $j \leqslant 2$, $m \in \omega$, and

(2.6) $b_2^n \nleqslant c_n$ if $n \notin A$.

Now (2.5) clearly guarantees that the t_j^m and b_j^m are all distinct, and then (2.6) shows that the recovery of A from any presentation of any partial ordering extending $\mathcal{P}_A^*$ must be correct.

The verification that $\mathcal{P}_A^*$ has TPP is like that for the $\mathcal{P}_n^*$. We must define the required sequence $Q_0, \{q_0\}, Q_1, \{q_1\}, \ldots,$ for each of (2.5) and (2.6). Suppose we are considering b_j^m on the left of the $\nleqslant$ relation. Fix $i < k$ such that $\{i, j, k\} = \{0, 1, 2\}$. We set $Q_0 = \{b_j^m, t_i^0\}$ unless $b_j^m \leqslant c_m$ in which case we let $Q_0 = \{b_j^m, t_i^0, a\}$. In either case $q_0 = t_k^0$. Suppose by induction Q_u and q_u have been

defined and that $q_u = t_v^0$ for $v \in \{j, k\}$. Let $\{v, v^*\} = \{j, k\}$. Define $Q_{u+1} = Q_u \cup \{b_v^{m-u-1}\}$ unless $b_v^{m-u-1} \leqslant c_{m-u-1}$ and $a \notin Q_u$. In this case let $Q_{u+1} = Q_u \cup \{b_v^{m-u-1}, a\}$. In either case set $q_{u+1} = t_{v^*}^0$. This procedure continues until Q_m is defined and so $m = n$ in the definition of TPP. It is now straightforward to check that the clauses of TPP are satisfied and we omit this verification.

Thus for any degree $\mathbf{a}$ and set $A \in \mathbf{a}$, $\mathcal{P}_A^*$ is embeddable in RED($\mathbf{a}$). Now if RED($\mathbf{a}$) $\cong$ RED($\mathbf{b}$) then $\mathcal{P}_A^*$ can be embedded in RED($\mathbf{b}$) and so A is recursive in the jump of any presentation of RED($\mathbf{b}$) as an upper semilattice. The standard presentation of RED($\mathbf{b}$) in terms of indices W_e^B for some fixed $B \in \mathbf{b}$ is recursive in $B^{(4)}$ and so $A \leqslant B^{(5)}$. Similarly $B \leqslant A^{(5)}$.

THEOREM 4. *If RED*($\mathbf{a}$) $\cong$ *RED*($\mathbf{b}$) *then* $\mathbf{a}$ *and* $\mathbf{b}$ *are contained in the same arithmetic degree. Indeed,* $\mathbf{a} \leqslant \mathbf{b}^{(5)}$.

The calculation that gives $\mathbf{a} \leqslant \mathbf{b}^{(5)}$ here can be improved by embedding more complicated partial orderings in RED($\mathbf{a}$). It is immediate from the proof of Lerman, Shore and Soare [**1981**] and the limit lemma that any $\mathbf{a}'$-presentable partial lattice having TPP can be embedded in RED($\mathbf{a}$). A bit more care would show that $\mathcal{P}_C^*$ is embeddable in RED($\mathbf{a}$) for any C which is Π_2 in A. Thus if RED($\mathbf{a}$) $\cong$ RED($\mathbf{b}$) then $\mathbf{a}^{(2)} \leqslant \mathbf{b}^{(5)}$.

COROLLARY 5. *The theory of RED*($\mathbf{a}$) *with added parameters is not recursive in* $\mathbf{a}$.

PROOF. Add on parameters corresponding to degrees $\mathbf{t}_0^0$, $\mathbf{t}_1^0$, $\mathbf{t}_2^0$ and $\mathbf{b}$ to generate $\mathcal{P}_B$ with $B \in \mathbf{a}'$.

COROLLARY 6. *The structure RED*($\mathbf{a}$) *is not presentable recursively in* $\mathbf{a}$ *even as a partial ordering.*

PROOF. If it were and some elements $\mathbf{t}_0^0$, $\mathbf{t}_1^0$, $\mathbf{t}_2^0$, $\mathbf{c}$ in it generate a partial lattice $\mathcal{P}_C$, then $C \leqslant \mathbf{a}'$ as we have argued. But we can in RED($\mathbf{a}$) generate $\mathcal{P}_C$ with $C \in \mathbf{a}''$ for a contradiction.

Turning now to the degrees as a whole, we first note that the recoverability of A from the jump of the presentation of any partial order in which $\mathcal{P}_A$ is embedded and the fact that one can easily extend $\mathcal{P}_A$ to a full lattice show that not every countable lattice can be embedded in, e.g., $\mathcal{D}[\mathbf{0}, \mathbf{0}']$, contrary to some speculations in Posner [**1980**, p. 59]. On the other hand, there are, as Posner there expects, methods of embedding simply presented lattices in $\mathcal{D}[\mathbf{0}, \mathbf{0}']$ much easier than the initial segments methods of Lerman [**1982**]. Indeed we can embed $\mathcal{P}_A$ or any $\mathbf{a}$-presentable lattice in $\mathcal{D}[\mathbf{a}, \mathbf{a}']$ using finite extension methods in the style of Kleene and Post [**1955**]. One also needs a simple representation theorem.

THEOREM 7. *Let* $\{p_i\}$ *enumerate an* $\mathbf{a}$*-presentable partial lattice* $\mathcal{P}$ (*i.e.* $\leqslant$, $\nleqslant$, $\vee$ *and* $\wedge$ *are partial recursive in* $\mathbf{a}$ *relations*) *for which* $\leqslant$ *and* $\nleqslant$ *define a partial order on* $\mathcal{P}$. *There is then a uniformly recursive in* $\mathbf{a}$ *array of functions* α_n *such that*

$$p_i \leqslant p_j \Leftrightarrow \forall n, m[\alpha_n(j) = \alpha_m(j) \rightarrow \alpha_n(i) = \alpha_m(i)], \tag{7.1}$$

$$p_i \vee p_j = p_k \Rightarrow \forall n, m[\alpha_n(i) = \alpha_m(i) \& \alpha_n(j) = \alpha_m(j) \rightarrow \alpha_n(k) = \alpha_m(k)], \tag{7.2}$$

(7.3)

$$p_i \wedge p_j = p_k \& \alpha_n(k) = \alpha_m(k) \Rightarrow$$
$$\exists q_1, q_2, q_3 \left[\alpha_n(i) = \alpha_{q_1}(i) \& \alpha_{q_1}(j) = \alpha_{q_2}(j) \& \alpha_{q_2}(i) = \alpha_{q_3}(i) \ \& \ \alpha_{q_3}(j) = \alpha_m(j) \right].$$

PROOF. This is essentially Jonsson **[1953]** translated into the language of Lerman **[1971]** or **[1982]** which we reproduce here. First we build a representation $\{\beta_n\}$ satisfying (7.1) and (7.2):

$$\beta_{2s+1}(i) = 2^{2s+1} \quad \forall i,$$
$$\beta_{2s}(i) = \begin{cases} 2^{2s+1} & \text{if } p_i \leqslant p_s, \\ 2^{2s} & \text{if } p_i \nleqslant p_s. \end{cases}$$

The check that (7.1) and (7.2) hold is routine.

For any situation as in the hypothesis of (7.3) we can add on the required three functions and still preserve (7.1) and (7.2):

Suppose $p_i \wedge p_j = p_k$ and $\gamma(k) = \delta(k)$. If $p_i \leqslant p_j$ then we can set $q_1 = q_2 = q_3 = \delta$. Otherwise we let w, x, y and z be new numbers not in the range of any of our functions. Now let

$$q_1(n) = \begin{cases} \gamma(n) & \text{if } p_n \leqslant p_i, \\ w & \text{if } p_n \nleqslant p_i; \end{cases}$$

$$q_2(n) = \begin{cases} \gamma(n) & \text{if } p_n \leqslant p_j, \\ x & \text{if } p_n \leqslant p_i \& p_n \nleqslant p_j, \\ y & \text{otherwise;} \end{cases}$$

$$q_3(n) = \begin{cases} \delta(n) & \text{if } p_n \leqslant p_j, \\ x & \text{if } p_n \leqslant p_i \& p_n \nleqslant p_j, \\ z & \text{otherwise.} \end{cases}$$

Again a straightforward check shows that (7.3) is now satisfied for this situation and (7.1) and (7.2) remain valid. We can therefore close off under this process to generate effectively in the presentation of $\mathcal{P}$ a representation with all the required properties.

Now suppose we are given an **a**-presentable partial lattice $\mathcal{P} = \{p_i\}$ that we wish to embed in $\mathcal{D}[\mathbf{a}, \mathbf{a}']$. We assume that $\leqslant$ and $\nleqslant$ give a complete partial order on $\mathcal{P}$ and so choose a representation $\{\alpha_n\}$ as in Theorem 7. We will build a function $g \leqslant \mathbf{a}'$ by specifying finite initial segments of g in a construction recursive in $\mathbf{a}'$. Our embedding $\mathcal{P} \to \mathcal{D}[\mathbf{a}, \mathbf{a}']$ will then be given by $p_i \mapsto a \vee h_i$ where $a \in \mathbf{a}$ and $h_i(n) = \alpha_{q(n)}(i)$. [If we wish to make the dependence of h_i on g explicit we write it as $h_i(g)$.] As $g \leqslant \mathbf{a}'$, $\deg(a \vee h_i) \in \mathcal{D}[\mathbf{a}, \mathbf{a}']$. Moreover if $p_i \leqslant p_j$ then $h_i \leqslant_T a \vee h_j$, for to compute $h_i(n) = \alpha_{g(n)}(i)$ we first compute $h_j(n) = \alpha_{g(n)}(j)$. We then find any m such that $\alpha_m(j) = h_j(n)$. By (7.1) $\alpha_m(i) = \alpha_{g(n)}(i) = h_i(n)$. We next note that if $p_i \vee p_j = p_k$ then $h_k \leqslant_T a \vee h_i \vee h_j$. To calculate $h_k(n)$ compute $h_i(n) = \alpha_{g(n)}(i)$ and $h_j(n) = \alpha_{g(n)}(j)$. Then find any m such that $\alpha_m(i) = h_i(m)$ and $\alpha_m(j) = h_j(n)$. By (7.2) $\alpha_m(k) = \alpha_{g(n)}(k) = h_k(n)$. Thus our embedding of $\mathcal{P}$ into $\mathcal{D}[\mathbf{a}, \mathbf{a}']$ will preserve $\leqslant$ and $\vee$. We must take steps in the construction of g to preserve $\nleqslant$ and $\wedge$ as well. We will write h_i for $a \vee h_i$ below.

Stage $s = \langle 0, e, i, j\rangle$. Suppose we have defined g^s with dom $g^s = t$. If $p_i \leqslant p_j$, we go to the next stage. Otherwise we let n and m witness that $p_i \nleqslant p_j$ as in (7.1). Ask (of $\mathbf{a}'$) if there is any finite extension g' of g^s such that $g'(t) = n$ and $\{e\}^{h_j(g')}(t)$ is convergent. If not let $g^{s+1} = g^s \cup \{\langle t, n\rangle\}$ and go on. If so we may choose such a g' with $\{e\}^{h_j(g')}(t) \neq \alpha_{g'(t)}(i)$ by switching the value of $g'(t)$ from n to m if necessary since this changes $\alpha_{g'(t)}(i)$ but does not affect $h_j(g')$ by our choice of n and m. We then let $g^{s+1} = g'$ and proceed.

Stage $s = \langle u, e, i, j\rangle$, $u > 0$. If $p_i \wedge p_j = p_k$ is not verified in u many steps, we go on. If it is, we ask (of $\mathbf{a}'$) if there are finite extensions g_1 and g_2 of g^s with dom g_1 = dom g_2 and

$$\alpha_{g_1(n)}(k) = \alpha_{g_2(n)}(k) \quad \forall n \in \text{dom } g_1,$$

and an x such that

$$\{e\}^{h_i(g_1)}(x) \neq \{e\}^{h_j(g_2)}(x).$$

If not we let $g^{s+1} = g^s$ and go on. If so we let the finite functions q_1, q_2 and q_3 extending g^s be as required by (7.3), i.e.

$$\alpha_{g_1(n)}(i) = \alpha_{q_1(n)}(i), \quad \alpha_{q_1(n)}(j) = \alpha_{q_2(n)}(j),$$
$$\alpha_{q_2(n)}(i) = \alpha_{q_3(n)}(i) \quad \text{and} \quad \alpha_{q_3(n)}(j) = \alpha_{g_2(n)}(j)$$

for every $n \in$ dom g_1. (We naturally get dom q_1 = dom q_2 = dom q_3 = dom g_1.) Thus $\{e\}^{h_i(g_1)}(x) = \{e\}^{h_i(q_1)}(x)$ and $\{e\}^{h_j(g_2)}(x) = \{e\}^{h_j(q_3)}(x)$.

We now ask if $\exists q_1'$ extending q_1 such that $\{e\}^{h_j(q_1')}(x)\downarrow$. If not let $g^{s+1} = q_1$ and go on. If so choose one. If $\{e\}^{h_i(q_1')}(x) \neq \{e\}^{h_j(q_1')}(x)$ let $g^{s+1} = q_1'$. If not we have $\{e\}^{h_i(g_1)}(x) = \{e\}^{h_i(q_1)}(x) = \{e\}^{h_i(q_1')}(x) = \{e\}^{h_j(q_1')}(x) = \{e\}^{h_j(q_2')}(x)$ where q_2' is the extension of q_2 by the same string added on to q_1 to get q_1', so that $h_j(q_1') = h_j(q_2')$ by our choice of q_1 and q_2. We now ask if there is an extension q_2'' of q_2' such that $\{e\}^{h_i(q_2'')}(x)\downarrow$. If not we let $g^{s+1} = q_2'$. If so we choose one. Again if $\{e\}^{h_i(q_2'')}(x) \neq \{e\}^{h_j(q_2'')}(x)$ we let $g^{s+1} = q_2''$. Otherwise we have $\{e\}^{h_i(g_1)}(x) = \{e\}^{h_j(q_2')}(x) = \{e\}^{h_j(q_2'')}(x) = \{e\}^{h_i(q_2'')}(x) = \{e\}^{h_i(q_3'')}(x)$ where q_3'' is the extension of q_3 gotten by adding on the same string that produces q_2'' from q_2 so that $h_i(q_2'') = h_i(q_3'')$ by our choice of q_2 and q_3. We now have $\{e\}^{h_i(q_3'')}(x) = \{e\}^{h_i(g_1)}(x) \neq \{e\}^{h_j(g_2)}(x) = \{e\}^{h_j(q_3)}(x) = \{e\}^{h_j(q_3'')}(x)$ and so we let $g^{s+1} = q_3''$ and proceed.

Our construction guarantees that $\nleqslant$ and $\wedge$ are preserved by the usual arguments. Thus we have proved the following.

THEOREM 8. *Any* **a**-*recursively presented partial lattice which is a partial ordering such as* $\mathcal{P}_A$ *can be embedded in* $\mathcal{D}[\mathbf{a}, \mathbf{a}']$.

One could of course improve on this result to embed more complicated lattices in $\mathcal{D}[\mathbf{a}, \mathbf{a}']$, but exactly which lattices are so embeddable is not clear. However this relatively simple result does imply that A is recursive in the jump of any presentation of $\mathcal{D}[\mathbf{a}, \mathbf{a}']$. Thus it can be used to replace coding by the much more difficult initial segment techniques in various arguments. For example, it immediately refutes the strong homogeneity conjecture by showing that if $\mathcal{D}[\mathbf{a}, \mathbf{a}'] = \mathcal{D}[\mathbf{b}, \mathbf{b}']$

then $\mathbf{a} \leqslant \mathbf{b}^{(5)}$ as $\mathcal{D}[\mathbf{b}, \mathbf{b}']$ is $\mathbf{b}^{(4)}$-presentable. Moreover, combined with the lemma of Harrington and Shore [**1981**] showing that $\mathcal{C}^{\mathbf{a}} = \{\mathbf{x} \geqslant \mathbf{a} | \exists \mathbf{y} \geqslant \mathbf{x}\ \forall \mathbf{z}(\mathbf{z} \vee \mathbf{y}$ is not a minimal cover of $\mathbf{z})\}$ is caught between $\mathcal{A}^{\mathbf{a}}$ and $\mathcal{H}^{\mathbf{a}}$ (the degrees above $\mathbf{a}$ of sets arithmetic and hyperarithmetic in $\mathbf{a}$ respectively), it refutes the homogeneity conjecture. Indeed it shows that if $\mathcal{D}(\geqslant \mathbf{a}) \cong \mathcal{D}(\geqslant \mathbf{b})$ then $\mathbf{a}$ and $\mathbf{b}$ are contained in the same hyperdegree.

For our final application of these codings, we note that by a variation on the usual permitting (or full approximation) methods the embedding of $\mathcal{P}_A$ into $\mathcal{D}[\mathbf{a}, \mathbf{a}']$ described above can be transformed to one into $\mathcal{D}[\mathbf{a}, \mathbf{c}]$ for any $\mathbf{c} > \mathbf{a}$ which is r.e. in $\mathbf{a}$.

The main problem is how to treat the multistep process used in the oracle construction at stages $s = \langle u, e, i, j \rangle$ with $u > 0$ when $p_i \wedge p_j = p_k$. The idea is to run several searches and permitting waits in a row. At an appropriate spot a (and all large enough stages s) we will first look for possible extensions g_1 and g_2 of $g^s \restriction a$ as described above. If we find them we can also immediately find q_1, q_2 and q_3 as required. We then will switch to q_1 when we are permitted to do so. If we are so permitted we begin searching for q_1' as described above. If we find it and are once again permitted, we switch to q_1' if it produces the desired disagreement (thereby assuring success) or to q_2' if it does not. In this last case, we start our final search for q_2''. If we find it and are again permitted, we switch to q_2'' if it produces the desired disagreement and otherwise to q_3''.

The start-up procedures at new points are much as usual if one thinks of each subsearch as making progress. If at point a we are looking for a certain extension but have not yet found it, no further activity is needed for this requirement at latter points. When we find it we start looking at some latter point b for such extensions until we are permitted at a. In this construction we may be permitted at a to make some progress which, rather than assuring success, requires another search. In this case we suspend work at b until this next extension is found for a (but is not yet permitted). One argues for the ultimate success of the construction by considering (after higher priority requirements have been satisfied) the last step in this process that we reach infinitely often but are never able to move beyond.

Consider now any 1-generic degrees $\mathbf{c}_1$, $\mathbf{c}_2$ as defined in Jockusch [**1980**]. By Theorem 5.1 of that paper $\mathbf{c}_i$ is r.e. in some $\mathbf{a}_i < \mathbf{c}_i$. Now $\mathcal{P}_{A_i}$ is embeddable in $\mathcal{D}[\mathbf{a}_i, \mathbf{c}_i]$ and so in $\mathcal{D}(\leqslant \mathbf{c}_i)$. Thus if $\mathcal{D}(\leqslant \mathbf{c}_1) \cong \mathcal{D}(\leqslant \mathbf{c}_2)$ then $\mathbf{a}_i \leqslant \mathbf{c}_j^{(4)}$ as $\mathcal{D}(\leqslant \mathbf{c}_i)$ is $\mathbf{c}_i^{(3)}$-presentable. Of course $\mathbf{c}_i \leqslant \mathbf{a}_i'$ and so we have the following.

THEOREM 9. *If* $\mathbf{c}_1$ *and* $\mathbf{c}_2$ *are* 1-*generic degrees* (*or even each one r.e. in some degree below it*) *and* $\mathcal{D}(\leqslant \mathbf{c}_1) \cong \mathcal{D}(\leqslant \mathbf{c}_2)$ *then they are contained in the same arithmetic degree. Indeed* $\mathbf{c}_1 \leqslant \mathbf{c}_2^{(5)}$.

As there are continuum many generic degrees we immediately get

COROLLARY 10. *There are generic* $\mathbf{c}_1$ *and* $\mathbf{c}_2$ *with* $\mathcal{D}(\leqslant \mathbf{c}_1) \not\cong \mathcal{D}(\leqslant \mathbf{c}_2)$.

This answers Question 8 of Jockusch [**1980**].

BIBLIOGRAPHY

L. Harrington and R. A. Shore, [1981]. *Definable degrees and automorphisms of* $\mathfrak{D}$, Bull. Amer. Math. Soc. (N.S.) **4**, 97–100.

C. G. Jockusch, Jr., [1980]. *Degrees of generic sets*, Recursion Theory: Its Generalizations and Applications (F. R. Drake and S. S. Wainer, eds.), London Math. Soc. Lecture Notes, no. 45, Cambridge Univ. Press, pp. 110–139.

B. Jonsson, [1953]. *On the representation of lattices*, Math. Scand. **1**, 193–206.

S. C. Kleene and E. L. Post, [1954]. *The upper semi-lattice of degrees of unsolvability*, Ann. of Math. (2) **59**, 379–407.

M. Lerman, [1971]. *Initial segments of the degrees of unsolvability*, Ann. of Math. (2) **93**, 365–389.

_____, [1982]. *Structure theory for the degrees of unsolvability*, Springer-Verlag, Berlin (to appear).

M. Lerman, R. A. Shore and R. I. Soare, [1981]. *The recursively enumerable degrees are not* $\aleph_0$*-categorical*, Adv. in Math. (to appear).

D. Posner, [1980]. *A survey of the non-RE degrees* $< \mathbf{0}'$, Recursion Theory: Its Generalizations and Applications (F. R. Drake and S. S. Wainer, eds.), London Math. Soc. Lecture Notes, no. 45, Cambridge Univ. Press, pp. 52–109.

G. E. Sacks, [1966]. *Degrees of unsolvability*, 2nd ed., Ann. of Math. Studies, no. 55, Princeton Univ. Press, Princeton, N.J.

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853