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EXTENSION OF A RESULT OF BEACHY AND BLAIR

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ABSTRACT. Let α: $R \to R$ be an automorphism of a ring R. If R is an α-reduced ring in which every faithful left ideal is cofaithful, then the same is true for the α-twisted power series ring $R^*[[x]]$.

Introduction. A module $_RM$ is said to be cofaithful if there exists a finite number of elements $m_1, \ldots, m_k$ in M with $\bigcap_{i=1}^{k} \operatorname{Ann}_R(m_i) = 0$. Writing M^k for the direct sum of k copies of M, it is clear that M is cofaithful if and only if there exists an exact sequence $0 \to R \to M^k$ in R-mod, where k is some integer ≥ 1. A result of Beachy and Blair [1] asserts that if R is a commutative ring in which every faithful ideal is cofaithful, then the same is true for the polynomial ring $R[x]$. The object of this note is to extend the validity of this result to skew power series rings under suitable assumptions on the ring R.

Throughout this note R will denote a ring with identity and all the modules considered will be unitary left modules.

1. α-reduced rings. Let α: $R \to R$ be a ring endomorphism satisfying $(1)\alpha = 1$. We write a^α for $(a)\alpha$ whenever $a \in R$. The twisted (or skew) power series ring corresponding to the endomorphism α will have as its elements the usual formal power series $\sum_{i\geq 0} x^i a_i$ with $a_i \in R$ ($x^0 = 1$ by convention) with addition defined in the usual way, but multiplication given by $x^i x^j = x^{i+j}$ and $a \cdot x = xa^\alpha$ for $i \geq 0$, $j \geq 0$ and $a \in R$. Then $a \cdot x^r = x^r a^{\alpha^r}$ where α^r: $R \to R$ is the rth power of α. We will fix an endomorphism α and denote the twisted power series ring corresponding to α by $R^*[[x]]$.

A ring R is said to be reduced if there are no nonzero nilpotent elements in R. If a, b are elements in a reduced ring, it is easily seen that $ab = 0 \Leftrightarrow ba = 0$.

DEFINITION 1.1. We say that R is α-reduced if R is reduced and further satisfies the condition $ab = 0 \Leftrightarrow a^\alpha b = 0 \Leftrightarrow ab^\alpha = 0$ for any a, b in R.

PROPOSITION 1.2. *Let R be α-reduced. Let $f = \sum_{i\geq 0} x^i a_i$, $g = \sum_{i\geq 0} x^i b_i$ be in $R^*[[x]]$. Then $fg = 0$ in $R^*[[x]]$ if and only if $a_i b_j = 0$ for all $i \geq 0, j \geq 0$.*

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PROOF. Let R be α-reduced and $a_i b_j = 0$ for all i, j. For any $k \geqslant 0$ the coefficient of x^k in fg is $\sum_{i+j=k;\, i\geqslant 0, j\geqslant 0} a_i^{\alpha^i} b_j$. Since R is α-reduced, from $a_i b_j = 0$ we immediately get $a_i^{\alpha} b_j = 0$, $a_i^{\alpha^2} b_j = 0, \ldots, a_i^{\alpha^r} b_j = 0$ for all $r \geqslant 0$. In particular $a_i^{\alpha^i} b_j = 0$. Hence $fg = 0$.

Conversely assume $fg = 0$ in $R^*[[x]]$. Then for any integer $k \geqslant 0$ we get

$$(\mathrm{S}_0) \qquad \sum_{\substack{i+j=k\\ i\geqslant 0, j\geqslant 0}} a_i^{\alpha^i} b_j = 0.$$

We will refer to the equation $\sum_{i+j=k;\, i\geqslant 0, j\geqslant 0} a_i^{\alpha^i} b_j = 0$ as the kth equation of the system (S_0). The 0th equation of (S_0) yields $a_0 b_0 = 0$. The first equation of (S_0) is

$$(\mathrm{A}_0) \qquad a_0^{\alpha} b_1 + \alpha_1 b_0 = 0.$$

Multiplying this on the left by b_0 we get

$$(\mathrm{B}_0) \qquad b_0 a_0^{\alpha} b_1 + b_0 a_1 b_0 = 0.$$

Since R is α-reduced, from $a_0 b_0 = 0$ we get $b_0 a_0 = 0$, $a_0^{\alpha^r} b_0 = 0 = a_0 a_0^{\alpha^r}$, $b_0^{\alpha^r} a_0 = 0 = b_0 a_0^{\alpha^r}$ for all $r \geqslant 1$. In particular $b_0 a_0^{\alpha} = 0$. Now, (B_0) yields $b_0 a_1 b_0 = 0$, which in turn yields $a_1 b_0 a_1 b_0 = 0$. Since R is reduced, this implies that $a_1 b_0 = 0$.

Assume inductively that we have proved that $a_i b_0 = 0$ for $0 \leqslant i \leqslant l$ (with $l \geqslant 1$). The $(l+1)$st equation in the system (S_0) is

$$(\mathrm{C}_0) \qquad a_0^{\alpha^{l+1}} b_{l+1} + a_1^{\alpha^l} b_l + \cdots + a_l^{\alpha} b_1 + a_{l+1} b_0 = 0.$$

Multiplying (C_0) on the left by b_0 we get

$$(\mathrm{D}_0) \qquad b_0 a_0^{\alpha^{l+1}} b_{l+1} + b_0 a_1^{\alpha^l} b_l + \cdots + b_0 a_l^{\alpha} b_1 + b_0 a_{l+1} b_0 = 0.$$

From $a_i b_0 = 0$ for $0 \leqslant i \leqslant l$ and the α-reducibility of R we get $b_0 a_i^{\alpha^r} = 0$ for $0 \leqslant i \leqslant l$ and all $r \geqslant 0$. Now, equation (D_0) yields $b_0 a_{l+1} b_0 = 0$. Hence $a_{l+1} b_0 a_{l+1} b_0 = 0$. Since R is reduced, we get $a_{l+1} b_0 = 0$.

(E_0) It follows that $a_i b_0 = 0$ for all $i \geqslant 0$.

Substituting (E_0) in (S_0), we get for every integer $k \geqslant 1$ the system of equations

$$(\mathrm{S}_1) \qquad \sum_{\substack{i+j=k\\ i\geqslant 0, j\geqslant 1}} a_i^{\alpha^i} b_j = 0.$$

We will refer to the equation $\sum_{i+j=k;\, i\geqslant 0, j\geqslant 1} a_i^{\alpha^i} b_j = 0$ as the kth equation of the system (S_1). The first equation of (S_1) yields $a_0^{\alpha} b_1 = 0$. The α-reduciblity of R now yields $a_0 b_1 = 0 = b_1 a_0$ and $b_1 a_0^{\alpha^r} = 0$ for any $r \geqslant 1$. The second equation of the system (S_1) is

$$(\mathrm{A}_1) \qquad a_0^{\alpha^2} b_2 + a_1^{\alpha} b_1 = 0.$$

Multiplying on the left by b_1 we get

$$(\mathrm{B}_1) \qquad b_1 a_0^{\alpha^2} b_2 + b_1 a_1^{\alpha} b_1 = 0.$$

Using $b_1 a_0^{\alpha^r} = 0$ for any $r \geqslant 1$ we see from (B_1) that $b_1 a_1^{\alpha} b_1 = 0$, which in turn yields $a_1^{\alpha} b_1 a_1^{\alpha} b_1 = 0$. The reduced nature of R now yields $a_1^{\alpha} b_1 = 0$. The α-reduced nature of R now yields $a_1 b_1 = 0$.

Assume inductively that we have proved that $a_i b_1 = 0$ for $0 \leq i \leq l$ (with $l \geq 1$). The α-reducibility of R then yields $b_1 a_i^{\alpha^r} = 0$ for $0 \leq i \leq l$ and $r \geq 0$. The $(l+2)$nd equation of the system (S_1) is

$$(C_1) \qquad a_0^{\alpha^{l+2}} b_{l+2} + a_1^{\alpha^{l+1}} b_1 + \cdots + a_{l+1}^{\alpha} b_1 = 0.$$

Multiplying (C_1) on the left by b_1 we get

$$(D_1) \qquad b_1 a_0^{\alpha^{l+2}} b_{l+2} + \cdots + b_1 a_{l+1}^{\alpha} b_1 = 0.$$

Using $b_1 a_i^{\alpha^r} = 0$ for $0 \leq i \leq l$ in (D_1) we get $b_1 a_{l+1}^{\alpha} b_1 = 0$. This in turn implies $a_{l+1}^{\alpha} b_1 a_{l+1}^{\alpha} b_1 = 0$ and the reduced nature of R implies $a_{l+1}^{\alpha} b_1 = 0$. The α-reduced nature of R now yields $a_{l+1} b_1 = 0$.

(E_1) It follows that $a_i b_1 = 0$ for all $i \geq 0$.

Using (E_1), the system (S_1) yields the system of equations

$$(S_2) \qquad \sum_{\substack{i+j=k \\ i\geq 0, j\geq 2}} a_i^{\alpha^j} b_j = 0 \quad \text{for any } k \geq 2.$$

Proceeding as before and using (S_2) we can show that

$$(E_2) \qquad a_i b_2 = 0 \quad \text{for all } i \geq 0.$$

Repeating this procedure we see that $a_i b_j = 0$ for all i, j.

2. The main result. We will now prove the main result of this paper.

THEOREM 2.1. *Let α: $R \to R$ be a ring automorphism of R with $(1)\alpha = 1$. Let R be α-reduced and $R^*[[x]]$ the α-twisted power series ring over R. If every faithful left ideal in R is cofaithful, then every faithful left ideal in $R^*[[x]]$ is cofaithful.*

PROOF. Let I be any faithful left ideal of $R^*[[x]]$. Let $I_0 = \{a \in R \mid$ there exists some $f \in I$ with a as one of the coefficients occurring in $f\}$. Let a, b be elements of I_0. Let $a =$ coefficient of x^i in f with $f \in I$ and $b =$ coefficient of x^j in g with $g \in I$. Then $a + b =$ coefficient of x^{i+j} in $x^j f + x^i g$ and $x^j f + x^i g \in I$. Hence $a + b \in I_0$.

Since α is an automorphism, α^{-1} exists. For any $r \in R$, the coefficient of x^i in $r^{\alpha^{-1}} \cdot f$ is precisely ra. Thus $a \in I_0$, $r \in R \Rightarrow ra \in I_0$. This proves that I_0 is a left ideal in R.

Let $r \in \mathrm{Ann}_R(I_0)$. Then for any $f \in I$, the coefficient of x^i in $r^{\alpha^{-1}} \cdot f = r \cdot$ (coefficient of x^i in f) $= 0$. Hence $r^{\alpha^{-1}} \cdot f = 0$ for any $f \in I$. Since I is faithful in $R^*[[x]]$, we see that $r^{\alpha^{-1}} = 0$. Hence $r = 0$. This proves that I_0 is faithful in R. Hence there exist finitely many elements $a_1, a_2, \ldots, a_k$ in I_0 with $\mathrm{Ann}_R(a_1, \ldots, a_k) = 0$.

Let $a_j =$ coefficient of x^{μ^j} in $f_j \in I$. Let $g = \sum_{i\geq 0} x^i \lambda_i$ be any element of $\mathrm{Ann}_{R^*[[x]]}(f_1, \ldots, f_k)$. From Proposition 1.2 we see that $\lambda_i a_j = 0$ for all $i \geq 0$ and $1 \leq j \leq k$. Since $\mathrm{Ann}_R(a_1, \ldots, a_k) = 0$ we get $\lambda_i = 0$ for all $i \geq 0$ and hence $g = 0$. This proves that $\mathrm{Ann}_{R^*[[x]]}(f_1, \ldots, f_k) = 0$, thereby showing that I is cofaithful in $R^*[[x]]$-mod.

COROLLARY 2.2. *If R is a commutative reduced ring in which every faithful ideal is cofaithful, then the same is true in the ordinary power series ring* $R^*[[x]]$.

PROOF. For a commutative reduced ring R, the conclusion of Proposition 1.2 is true. Hence the proof of Theorem 2.1 goes through when R is commutative.

REFERENCES

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