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UNITARY ONE-PARAMETER GROUPS WITH FINITE SPEED OF PROPAGATION

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ABSTRACT. Suppose that ξ is a Hermitian vector bundle over a Riemannian manifold and that U is a one-parameter group of linear operators on the set of smooth sections of ξ with compact support. We prove that if U satisfies a smoothness condition, is unitary, and propagates initial data with finite speed, then it can be constructed from the solutions of a first-order symmetric hyperbolic system of partial differential equations.

Suppose that ξ is a Hermitian vector bundle over a Riemannian manifold M. In [2] Chernoff studied first-order symmetric hyperbolic systems of partial differential equations associated with ξ. He proved that if M is complete and the coefficients of such a system are time-independent, smooth, and sufficiently small, then the solutions of the system can be used to construct a one-parameter group of linear operators on the set of smooth sections of ξ with compact support. He also proved that under these conditions this one-parameter group has three properties: it satisfies a smoothness condition, is unitary, and propagates initial data with finite speed.

In this paper we prove a converse to Chernoff's results: any one-parameter group of linear operators on the set of smooth sections of ξ with compact support with these three properties can be constructed from the solutions of a first-order symmetric hyperbolic system belonging to a natural class of such systems. If M is complete, this result, together with Chernoff's results, characterizes the systems belonging to this class.

We will use local techniques to prove our result. Fourier transform techniques (such as those developed in [3] by Gel′fand and Šilov) can be used to prove results like ours for translation-invariant one-parameter groups on spaces of functions on R^n or other Lie groups; in [1] Berman used such techniques to characterize certain translation-invariant second-order wave equations. (Our techniques can also be used to characterize such higher-order equations.)

Before stating our result more precisely, we give a number of preliminaries:

Suppose that c is a nonnegative real number (the upper limit on the speed of propagation). Also, suppose that M is a C^∞ manifold with a C^∞ Riemannian metric $\langle -, -\rangle_M$ and that ξ is a complex C^∞ vector bundle over M with a C^∞ Hermitian structure $\langle -, -\rangle_\xi$ (inner product on each fiber ξ_x of ξ).

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Let $C_0^\infty(\xi)$ denote the set of C^∞ sections of ξ with compact support, and let $\langle -, - \rangle$ denote the inner product on $C_0^\infty(\xi)$ defined by

$$\langle f, g \rangle = \int_M \langle f, g \rangle_\xi \, dV,$$

where dV is the measure associated with the Riemannian metric.

We now define the class $\mathfrak{U}(c)$ of one-parameter groups appearing in our result: Suppose that U is a one-parameter group of linear operators on $C_0^\infty(\xi)$. If f is in $C_0^\infty(\xi)$, let Uf be the function from $M \times R$ into ξ defined by

$$Uf(x, t) = U_t f(x).$$

Say that U is smooth if Uf is C^∞ for all f in $C_0^\infty(\xi)$. And say that U propagates initial data with speed c or less if, for all f in $C_0^\infty(\xi)$ and t in R, the distance between the support of f and any point in the support of $U_t f$ is less than or equal to $c|t|$. Let $\mathfrak{U}(c)$ denote the set of smooth unitary one-parameter groups of linear operators on $C_0^\infty(\xi)$ that propagate initial data with speed c or less.

If U is a smooth one-parameter group of linear operators on $C_0^\infty(\xi)$, let dU, the infinitesimal generator of U, be the linear operator on $C_0^\infty(\xi)$ defined by

$$dUf(x) = \frac{\partial Uf}{\partial t}(x, 0).$$

Our result is in terms of the operators dU; we now define the class $\mathfrak{L}(c)$ of such operators appearing in it: Suppose that L is a zeroth- or first-order linear differential operator on $C_0^\infty(\xi)$. In [2] Chernoff defined the speed of propagation $c_L(x)$ of L at a point x in M as follows:

$$c_L(x) = \sup\{ \|\sigma_L(u, x)\|_{\mathrm{op}} : u \in T_x^* M, \|u\|_{\mathrm{cs}} = 1 \},$$

where σ_L is the symbol of L, $\|-\|_{\mathrm{op}}$ is the operator norm, and $\|-\|_{\mathrm{cs}}$ is the Riemannian norm on the cotangent space T_x^*M. Let $\mathfrak{L}(c)$ denote the set of formally skew-adjoint zeroth- or first-order linear differential operators on $C_0^\infty(\xi)$ such that $c_L(x) \leq c$ for all x in M.

We can now state our result:

THEOREM 1. $d(\mathfrak{U}(c)) \subset \mathfrak{L}(c)$.

This result implies that if U is in $\mathfrak{U}(c)$, then there is a first-order symmetric hyperbolic system

$$\partial \tilde{f} / \partial t = L \tilde{f} \tag{1}$$

with L in $\mathfrak{L}(c)$ such that if f is in $C_0^\infty(\xi)$, then Uf is the unique C^∞ solution of (1) satisfying the initial condition $\tilde{f}(x, 0) = f(x)$. ($L = dU$, of course.)

In [2] Chernoff obtained results that imply that if M is complete, then $d(\mathfrak{U}(c)) \supset \mathfrak{L}(c)$. Thus we have the following

COROLLARY. *If M is complete, then* $d(\mathfrak{U}(c)) = \mathfrak{L}(c)$.

If U_1 and U_2 are in $\mathfrak{U}(c)$ and $dU_1 = dU_2$, then $U_1 = U_2$ (as one can easily show); this fact and the corollary imply that if M is complete, then the map $U \to dU$ establishes a one-to-one correspondence between $\mathfrak{U}(c)$ and $\mathfrak{L}(c)$.

Before proving Theorem 1, we discuss the nature of dU when U is not in $\mathfrak{U}(c)$. Suppose that U is a smooth one-parameter group of linear operators on $C_0^\infty(\xi)$ (so that dU is defined). If U propagates initial data with speed c or less for some c, then dU is a differential operator, as our proof of Theorem 1 will show. However, if U does not propagate initial data with speed c or less for any c, then dU might not be a differential operator. For example, suppose that M is R with its usual Riemannian metric, that ξ is the product vector bundle $R \times C^2$ over R with its usual Hermitian structure, and that a is a nonzero real number; now define U by

$$U_t(f_1, f_2)(x) = (f_1(x), f_2(x) + tf_1(x + a)).$$

Then U does not propagate initial data with speed c or less for any c, and dU is not a differential operator. And even if U does propagate initial data with speed c or less for some c (so that dU is a differential operator), dU might not have order zero or one. For example, suppose that M and ξ are as above and that k is an integer greater than one; now define U by

$$U_t(f_1, f_2)(x) = \big(f_1(x), f_2(x) + tf_1^{(k)}(x)\big).$$

Then U propagates initial data with speed zero, but dU has order greater than one (k, in fact). (U is not unitary, of course.)

Proof of Theorem 1. Suppose that U is in $\mathfrak{U}(c)$, and let $L = dU$. Since U propagates initial data with speed c or less, L does not increase supports. According to a theorem of Peetre [5], L is therefore a differential operator. The formal skew-adjointness of L can be proved in the usual way (by differentiating the equation expressing the unitarity of U with respect to t and setting t equal to zero).

Our proof uses a domain-of-dependence inequality involving integrals over closed balls $B(a, \tilde{r})$ with center a and radius $\tilde{r}$. Suppose that $r > 0$ and that f is in $C_0^\infty(\xi)$, and let

$$I(t) = \int_{B(a, r + c|t|)} \|U_t f\|_\xi^2 \, dV,$$

where $\|-\|_\xi$ is the norm associated with $\langle -, - \rangle_\xi$; the inequality we need is

$$I(t) \geqslant I(0). \tag{2}$$

This can be proved using a sequence φ_k of C^∞ cut-off functions. More specifically, suppose that φ_k is identically equal to 1 on $B(a, r + c|t|)$ and is identically equal to 0 on $B(a, r + c|t| + \frac{1}{k})$. Then $\varphi_k U_t f$ and $U_t f$ agree on $B(a, r + c|t|)$, and

$$\int_M \|\varphi_k U_t f\|_\xi^2 \, dV \to I(t). \tag{3}$$

Since U propagates initial data with finite speed or less, $U_{-t}(\varphi_k U_t f)$ and $U_{-t}(U_t f) = f$ agree on $B(a, r)$. Therefore

$$\int_M \|U_{-t}(\varphi_k U_t f)\|_\xi^2 \, dV \geqslant I(0). \tag{4}$$

(2) follows from (3), (4), and unitarity.

Suppose that $B(a, r)$ is contained in a normal neighborhood of a [4, p. 33]. Then $I'(0+)$ exists and is given by

$$I'(0+) = c\int_{S(a,r)} \|f\|_\xi^2 \, dS + \int_{B(a,r)} (\langle Lf, f\rangle_\xi + \langle f, Lf\rangle_\xi) \, dV,$$

where $S(a, r)$ is the sphere with center a and radius r, and dS is the measure on $S(a, r)$ associated with the Riemannian metric on it induced by $\langle -, -\rangle_M$. (2) implies that $I'(0+) \geqslant 0$. Therefore

$$-\int_{B(a,r)} (\langle Lf, f\rangle_\xi + \langle f, Lf\rangle_\xi) \, dV \leqslant c\int_{S(a,r)} \|f\|_\xi^2 \, dS. \tag{5}$$

We will prove that (5) is violated if L is not in $\mathfrak{L}(c)$. First we write (5) in local coordinates. Suppose that b is on $S(a, r)$ and that Ω is a neighborhood of b. Also, suppose that $x_1, \ldots, x_n$ are local coordinates for Ω, with x_1 the distance from a. Let m be the order of L, and suppose that in these coordinates $L = A_1 D_1^m + \ldots,$ where the dotted terms contain only lower powers of D_1. (Given b, we can always choose a, r, Ω, and $x_1, \ldots, x_n$ in such a way that these suppositions are satisfied.) Integration by parts shows that the formal adjoint $L^\dagger$ of L is given by

$$L^\dagger = (-1)^m A_1^\dagger D_1^m + \ldots.$$

Note that

$$A_1^\dagger = (-1)^{m+1} A_1 \tag{6}$$

(because $L^\dagger = -L$). Suppose that the support of f is contained in Ω. Then in our coordinates (5) becomes

$$-\int_{x_1 < r} ((A_1 D_1^m f)^\dagger f + f^\dagger (A_1 D_1^m f) + \ldots) \, dV \leqslant c\int_{x_1 = r} |f|^2 \, dS. \tag{7}$$

(Let g be the determinant of the metric tensor; then dV and dS are $g^{1/2}$ times Lebesgue measure on R^n and R^{n-1}, respectively.) Now we integrate the left side of (7) by parts and then use the formal skew-adjointness of L; we find that when $m > 1$,

$$-\int_{x_1 = r} (2 \operatorname{Re}(f^\dagger A_1 D_1^{m-1} f) + \ldots) \, dS \leqslant c\int_{x_1 = r} |f|^2 \, dS \tag{8}$$

(where again the dotted terms contain only lower powers of D_1), and that when $m = 1$,

$$-\int_{x_1 = r} f^\dagger A_1 f \, dS \leqslant c\int_{x_1 = r} |f|^2 \, dS. \tag{9}$$

(When $m = 0$, the left side of (7) is equal to zero.)

We will use (8) and (9) to complete our proof. First suppose by way of contradiction that $m > 1$. Take a point b in M at which the order of L is exactly m. Then choose a, r, Ω, and $x_1, \ldots, x_n$ in such a way that $A_1(b) \neq 0$. But then we can construct an f that violates (8). Therefore $m \leqslant 1$. Next suppose that $m = 1$. Take any point b in M and any unit vector u in $T_b^* M$. Then choose a, r, Ω, and

$x_1, \ldots, x_n$ in such a way that the local coordinates of u are $(1, 0, \ldots, 0)$. (We can always do this because x_1 is the distance from a.) With these assumptions,

$$\sigma_L(u, b) = A_1(b).$$

(6) implies that $A_1(b)$ is selfadjoint and hence has only real eigenvalues. The magnitudes of these eigenvalues must be less than or equal to c because otherwise we could construct an f that violates (9). Therefore

$$\|\sigma_L(u, b)\|_{\mathrm{op}} = \|A_1(b)\|_{\mathrm{op}} \leqslant c,$$

$c_L(b) \leqslant c$, and L is in $\mathfrak{L}(c)$. Finally, suppose that $m = 0$. Then σ_L and hence c_L are identically zero, and therefore L is in $\mathfrak{L}(c)$.

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