PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 3, March 1982

PROPERTY L AND ASYMPTOTIC ABELIANNESS FOR VON NEUMANN ALGEBRAS OF TYPE I

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ABSTRACT. We prove a correct assertion on Property L for von Neumann algebras of type I: a type I von Neumann algebra M on a separable Hilbert space has Property L if and only if M contains no minimal projection. Furthermore, a correct proof of an assertion on asymptotic abelianness for von Neumann algebras of type I is also given.

Let M be a von Neumann algebra on a separable Hilbert space and let $U(M)$ represent the set of unitary operators in M. A uniformly bounded sequence $\{A_n\}$ in M is said to be central if $\{AA_n - A_nA\}$ converges strongly to zero for each A in M. Recall that M is said to have Property L if there is a central sequence $\{U_n\}$ in $U(M)$ such that weak limit $U_n = 0$. Moreover M is said to be asymptotically abelian if there is a sequence of *-automorphisms $\{\alpha_n\}$ on M such that $\{\alpha_n(A)\}$ is central for each A in M.

In [2, Theorem 3], Sarian stated that type I von Neumann algebras do not have Property L. However, this theorem does not hold in general. Indeed, we can easily give a counter example. Let T be the unit circle and $M_{L^\infty(T)}$ be the algebra of all multiplications $\pi(f)$ on $L^2(T)$ by essentially bounded Lebesgue measurable functions f on T. We put $U_n(t) = e^{itn}$ $(0 \leqslant t \leqslant 2\pi)$. Then the sequence $\{\pi(U_n)\}$ in $U(M_{L^\infty(T)})$ converges weakly to zero by the Riemann-Lebesgue Theorem. Since $M_{L^\infty(T)}$ is abelian, $\{\pi(U_n)\}$ is, of course, central. Thus $M_{L^\infty(T)}$ is a type I von Neumann algebra which has Property L. In the present paper, we give a necessary and sufficient condition for von Neumann algebras of type I to have Property L.

In [3, Corollary 6], Sarian showed that if M is a finite type I von Neumann algebra having no abelian direct summand, then M is not asymptotically abelian. His proof, however, depends on the implication that if M is asymptotically abelian then it has Property L, which cannot be applied to the present case because of the above situation. Hence we have, independent of Property L, that nonabelian von Neumann algebras of type I are not asymptotically abelian.

1. Property L **for von Neumann algebras of type** I. We note first the relation between Property L and typical von Neumann algebras of type I: $M_{L^\infty(T)}$ and $B(H)$ (the full operator algebra on a separable Hilbert space H). As we have stated in the introduction, we have the following.

Received by the editors April 3, 1981 and, in revised form, May 6, 1981.
1980 *Mathematics Subject Classification*. Primary 46L10.
Key words and phrases. von Neumann algebra, type I, Property L, minimal projection, asymptotic abelianness.

LEMMA 1.1. *$M_{L^\infty(T)}$ has Property* L.

It is known that $B(H)$ does not have Property L. Indeed, under the assumption that M and N are factors, Tomiyama proved that if either M or N has Property L then the tensor product $M \otimes N$ has Property L [5, Theorem 2]. Hence, if one supposes that $B(H)$ has Property L, then $B(H) \otimes M$ has Property L for any factor M of type III, which we find impossible by considering the example of Pukánszky [1] for M. Besides, a direct proof of this fact was given by Willig [6].

LEMMA 1.2. *$B(H)$ does not have Property* L.

Tomiyama's Theorem mentioned above holds without the assumption that M and N are factors, because we can apply his technique to the general case. Thus the following lemma is a special case of his theorem, but we give a short proof directly.

LEMMA 1.3. *Let $\mathcal{Q}$ be an abelian von Neumann algebra which has Property* L. *Then, for any von Neumann algebra M on a separable Hilbert space H, the tensor product $\mathcal{Q} \otimes M$ has Property* L.

PROOF. Let $\{u_n\}$ be a sequence of unitaries in $\mathcal{Q}$ such that weak limit $u_n = 0$. We put $U_n = u_n \otimes 1$ in $\mathcal{Q} \otimes M$. Then we have weak limit $U_n = 0$ in $\mathcal{Q} \otimes \mathbf{C}1$ and also in $\mathcal{Q} \otimes M$. Since $\mathcal{Q} \otimes \mathbf{C}1$ is contained in the center of $\mathcal{Q} \otimes M$, $\{U_n\}$ is central in $\mathcal{Q} \otimes M$. Q.E.D.

The following lemma is easily derived from the definitions.

LEMMA 1.4. *The direct sum $\sum \oplus M_\alpha$ of von Neumann algebras has Property* L *if and only if each M_α has Property* L.

We here give a necessary and sufficient condition for a von Neumann algebra of type I to have Property L.

THEOREM 1.5. *Let M be a type* I *von Neumann algebra on a separable Hilbert space H. Then M has Property* L *if and only if M contains no minimal projection.*

PROOF. We note first that a von Neumann algebra M of type I is *-isomorphic to the direct sum

$$\sum \oplus (\mathcal{Q}_\alpha \otimes B(H_\alpha))$$

where each $\mathcal{Q}_\alpha$ is an abelian von Neumann algebra on a separable Hilbert space and $\dim H_\alpha = \alpha$ [4, V, Theorem 1.27]. Suppose M contains a minimal projection. Then M has a direct summand $\mathcal{Q}_\alpha \otimes B(H_\alpha)$ which contains a minimal projection. Let P be a minimal projection in $\mathcal{Q}_\alpha \otimes B(H_\alpha)$, and we denote by $Z(P)$ the central support of P in it, so that $Z(P)$ is of the form $Z(P) = q \otimes 1$ where q is a minimal projection in $\mathcal{Q}_\alpha$. Thus $q\mathcal{Q}_\alpha \otimes B(H_\alpha)$ is *-isomorphic to $B(H_\alpha)$ and, since $\mathcal{Q}_\alpha$ is abelian, we have

$$\mathcal{Q}_\alpha \otimes B(H_\alpha) = (q\mathcal{Q}_\alpha \otimes B(H_\alpha)) \oplus ((1-q)\mathcal{Q}_\alpha \otimes B(H_\alpha)).$$

Hence, by Lemma 1.2, M has a direct summand which does not have Property L. Therefore M does not have Property L by Lemma 1.4.

Conversely we suppose that M contains no minimal projection. Then no minimal projection is contained in each direct summand $\mathcal{Q}_\alpha \otimes B(H_\alpha)$, thus each $\mathcal{Q}_\alpha$ contains no minimal projection. Since $\mathcal{Q}_\alpha$ acts on a separable Hilbert space, $\mathcal{Q}_\alpha$ is *-isomorphic to $M_{L^\infty(T)}$ [4, III. Theorem 1.22]. By Lemma 1.1, 1.3 and 1.4, M has Property L. Q.E.D.

2. Asymptotic abelianness for von Neumann algebras of type I. Let M_n be a factor of type I_n $(2 \leq n < \infty)$. We regard M_n as the full operator algebra $B(H)$ on the n-dimensional Euclidean space H. Let $\{e_i\}_{i=1}^n$ be an orthogonal basis of H and $\{e_{ij}\}_{i,j=1}^n$ be the corresponding basis for $B(H)$, that is, $e_{ij}^* e_{ij} = P_j$ and $e_{ij}e_{ij}^* = P_i$ where P_i is the projection onto the subspace spanned by e_i.

LEMMA 2.1. *Let $\mathcal{Q}$ be an abelian von Neumann algebra. Then the tensor product $\mathcal{Q} \otimes M_n$ $(2 \leq n < \infty)$ is not asymptotically abelian.*

PROOF. Suppose $\{\alpha_k\}$ is a sequence of *-automorphisms on $\mathcal{Q} \otimes M_n$ such that $\{\alpha_k(A)\}$ is central for each A in $\mathcal{Q} \otimes M_n$. Since any *-automorphism of a von Neumann algebra maps its center onto itself, there is a *-automorphism β_k of $\mathcal{Q}$, corresponding to each α_k, such that $\alpha_k = \beta_k \otimes 1$ on $\mathcal{Q} \otimes \mathbf{C}1$. We put $\gamma_k = \alpha_k(\beta_k^{-1} \otimes 1)$. Then γ_k is a *-automorphism on $\mathcal{Q} \otimes M_n$ leaving the center elementwise fixed. Hence γ_k is inner (cf. [4, V. §1, Exercise 4]). Moreover $\{\gamma_k\}$ satisfies the property of asymptotic abelianness. In fact, for f and g in $\mathcal{Q}$, we have

$$
\begin{aligned}
&\gamma_k(f \otimes e_{ij}) \cdot g \otimes e_{pq} - g \otimes e_{pq} \cdot \gamma_k(f \otimes e_{ij}) \\
&\qquad = f \otimes 1 \cdot \alpha_k(1 \otimes e_{ij}) \cdot g \otimes e_{pq} - g \otimes e_{pq} \cdot f \otimes 1 \cdot \alpha_k(1 \otimes e_{ij}) \\
&\qquad = fg \otimes 1 \cdot \left(\alpha_k(1 \otimes e_{ij}) \cdot 1 \otimes e_{pq} - 1 \otimes e_{pq} \cdot \alpha_k(1 \otimes e_{ij})\right) \\
&\qquad \to 0 \quad \text{(strongly)}.
\end{aligned}
$$

Hence, for any pair $A = \sum_{i,j=1}^n f_{ij} \otimes e_{ij}$ and $B = \sum_{i,j=1}^n g_{ij} \otimes e_{ij}$ in $\mathcal{Q} \otimes M_n$, we have

$$
\begin{aligned}
\gamma_k(A)B - B\gamma_k(A) &= \sum_{i,j=1}^n \gamma_k(f_{ij} \otimes e_{ij}) \cdot \sum_{p,q=1}^n g_{pq} \otimes e_{pq} \\
&\quad - \sum_{p,q=1}^n g_{pq} \otimes e_{pq} \cdot \sum_{i,j=1}^n \gamma_k(f_{ij} \otimes e_{ij}) \\
&= \sum_{i,j=1}^n \sum_{p,q=1}^n \left(\gamma_k(f_{ij} \otimes e_{ij}) \cdot g_{pq} \otimes e_{pq} - g_{pq} \otimes e_{pq} \cdot \gamma_k(f_{ij} \otimes e_{ij})\right) \\
&\to 0 \quad \text{(strongly)}.
\end{aligned}
$$

Let $\natural$ denote the canonical central trace on $\mathcal{Q} \otimes M_n$ (cf. [4, V, Proposition 1.23]). Then, for each element of the form $A = 1 \otimes a$ $(a \in M_n)$, we have $A^\natural = \tau(a)1 \otimes 1$ where τ is the normalized trace on M_n. Since γ_k is inner, the map $\natural$ is γ_k-invariant for each k [4, V, Theorem 2.6]. We shall show that, for each a in $\mathcal{Q}$, $\{\gamma_k(1 \otimes a)\}$ converges strongly to $\tau(a)1 \otimes 1$. We put, for each k,

$$\gamma_k(1 \otimes a) = \sum_{i,j=1}^{n} f_{ij}^{(k)} \otimes e_{ij} \qquad (f_{ij}^{(k)} \in \mathcal{Q}).$$

Then we have

$$\tau(a)1 \otimes 1 = (1 \otimes a)^{\natural} = (\gamma_k(1 \otimes a))^{\natural} = (1/n)\left(\sum_{i=1}^{n} f_{ii}^{(k)}\right) \otimes 1,$$

that is,

$$\tau(a)1 = (1/n) \sum_{i=1}^{n} f_{ii}^{(k)} \quad \text{for all } k.$$

Since the sequence $\{\gamma_k(1 \otimes a)\}$ is central, for any A, B and C in $\mathcal{Q} \otimes M_n$, it follows that

$$(*) \qquad A(B\gamma_k(1 \otimes a) - \gamma_k(1 \otimes a)B)C \to 0 \quad \text{(strongly)}.$$

If we put $A = C = 1 \otimes e_{ij}$ and $B = 1 \otimes e_{ji}$ $(i \neq j)$ in $(*)$, we have

$$\left(f_{ii}^{(k)} - f_{jj}^{(k)}\right) \otimes e_{ij} \to 0 \quad \text{(strongly) as } k \to \infty.$$

Thus $\{f_{ii}^{(k)} - f_{jj}^{(k)}\}$ converges strongly to zero for $i \neq j$. Therefore we have

$$nf_{ii}^{(k)} = (n-1)f_{ii}^{(k)} - \sum_{j \neq i} f_{jj}^{(k)} + \sum_{j=1}^{n} f_{jj}^{(k)} \to n\tau(a)1 \quad \text{(strongly)}.$$

On the other hand, putting $A = 1 \otimes e_{ii}$, $B = 1 \otimes e_{ji}$ and $C = 1 \otimes e_{ij}$ $(i \neq j)$ in $(*)$, we have

$$f_{ij}^{(k)} \otimes e_{ij} \to 0 \quad \text{(strongly) as } k \to \infty.$$

Consequently it follows that

$$\begin{aligned} \gamma_k(1 \otimes a) &= \sum_{i,j=1}^{n} f_{ij}^{(k)} \otimes e_{ij} \to \sum_{i=1}^{n} \tau(a)1 \otimes e_{ii} \quad \text{(strongly)} \\ &= \tau(a)1 \otimes 1. \end{aligned}$$

Hence, for any a and b in M_n, $\{\gamma_k(1 \otimes ab)\}$ converge strongly to $\tau(ab)1 \otimes 1$. Since $\{\gamma_k(1 \otimes b)\}$ is bounded, we have

$$\gamma_k(1 \otimes ab) = \gamma_k(1 \otimes a)\gamma_k(1 \otimes b) \to \tau(a)\tau(b)1 \otimes 1 \quad \text{(strongly)}.$$

Namely $\tau(ab) = \tau(a)\tau(b)$ on M_n. But this is a contradiction. Q.E.D.

We have now given a correct proof of the fact that von Neumann algebras of type I_n $(2 \leqslant n < \infty)$ are not asymptotically abelian. Sarian proved, independent of Property L, that von Neumann algebras of type I_∞ are not asymptotically abelian [3, Theorem 8]. Hence we established his theorem [3, Theorem 11].

THEOREM 2.2 [3, THEOREM 11]. *Nonabelian type* I *von Neumann algebras are not asymptotically abelian.*

ACKNOWLEDGEMENT. The author would like to thank Professor J. Tomiyama (Niigata Univerity) for useful discussion with him.

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