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FIXED POINTS OF NONEXPANSIVE CONDENSING MULTI-VALUED MAPPINGS ON METRIC SPACES

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ABSTRACT. In this paper we consider the existence of fixed points of nonexpansive condensing multi-valued mappings from a certain kind of metric space into itself; the spaces, here, are neither linear nor compact. Our result generalizes a theorem of Dotson and also a theorem of Bose and Mukherjee in some respect.

In this note we consider the existence of fixed points of nonexpansive condensing multi-valued mappings from a certain kind of metric space into itself. This result generalizes a theorem of Dotson [**2**] and also a theorem of Bose and Mukherjee [**1**] in some respect.

Let (X, d) be a metric space. Denote by 2^X the set of all closed subsets of X and by H the Hausdorff metric on 2^X induced by metric d.

DEFINITION 1. If $F = \{f_A \mid A \in 2^X\}$ is a family of functions from $[0, 1]$ into 2^X satisfying the following conditions:

(I) $f_A(1) = A$ for all $A \in 2^X$,

(II) there is a function ϕ: $[0, 1) \to [0, 1)$ with

$$H(f_A(t), f_B(t)) \leqslant \phi(t)H(A, B)$$

for all A, B in 2^X and $t \in [0, 1)$,

(III) for all $A \in 2^X$, all $t_0 \in [0, 1]$,

$$\lim_{t \to t_0} H(f_A(t), f_A(t_0)) = 0,$$

and for all $t \in [0, 1]$,

(IV) $f_A(t) \subset f_B(t)$ for all A, B in 2^X with $A \subset B$, and

(V) $f_A(t)$ is compact whenever the set A is finite;

we call F an (S)-convex structure on X.

REMARK 1. If X is a starlike convex subset of a Banach space and if x_0 is a starlike center of X, then for each $A \subset X$, we define

$$f_A(t) = (1 - t)x_0 + tA, \qquad t \in [0, 1].$$

$F = \{f_A \mid A \in 2^X\}$ is an (S)-convex structure.

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Let γ denote the measure of noncompact sets on X [4], that is, for each bounded subset A of X,

$$\gamma(A) = \inf\{r > 0 \mid A \text{ can be covered by finitely many balls of radii} \leqslant r\}.$$

DEFINITION 2. Let $T\colon X \to 2^X$.

(1) T is said to be nonexpansive if for any $x \in X$ and any $y \in X$, we have

$$H(Tx, Ty) \leqslant d(x, y).$$

(2) A continuous mapping T of (X, d) into $(2^X, H)$ is said to be condensing if for any bounded set $A \subset X$, we have

$$\gamma(T(A)) < \gamma(A)$$

provided that $\gamma(A) \neq 0$.

Our result is the following:

THEOREM. *Let (X, d) be a bounded complete metric space with an (S)-convex structure F. Then every nonexpansive condensing multi-valued mapping T of X into 2^X has a fixed point.*

Before we prove our theorem, we first need some preliminary results related to measure of noncompact sets as follows:

LEMMA 1. *Let γ be defined as before. Then*

(1) $\gamma(A) = 0$ *if and only if* $\bar{A}$ *is compact,*

(2) $\gamma(\bigcup_{i=1}^{n} A_i) = \max\{\gamma(A_i) \mid 1 \leqslant i \leqslant n\}$,

(3) $\gamma(A) \leqslant \gamma(B)$ *whenever* $A \subset B$,

(4) $\gamma((A)_r) \leqslant \gamma(A) + r$, *where* $r \geqslant 0$ *and* $(A)_r = \{b \in X \mid d(b, A) \leqslant r\}$,

(5) $\gamma(\bar{A}) = \gamma(A)$ *for all bounded subsets* A *of* X.

The proof of Lemma 1 is known (see Sadovskii [5]).

REMARK 2. It is easy to see that $r > \gamma(A)$ if and only if there is a finite set G such that $(G)_r \supset A$ if and only if there is a finite set J such that $H(J, A) \leqslant r$.

In order to prove our theorem we need the following three lemmas.

LEMMA 2. *Suppose that F satisfies* (II) *and* (III) *of Definition* 1. *Then for* $\{A_n \mid n = 0, 1, 2, \ldots\} \subset 2^X$ *and* $\{t_n \mid n = 0, 1, 2, \ldots\} \subset [0, 1]$ *with* $\lim_{n\to\infty} H(A_n, A_0) = 0$ *and* $\lim_{n\to\infty} t_n = t_0$, *we have*

$$\lim_{n\to\infty} H\big(f_{A_n}(t_n), f_{A_0}(t_0)\big) = 0.$$

PROOF. Since the triangle inequality is valid for the Hausdorff metric and (II) we have

$$\begin{aligned} H\big(f_{A_n}(t_n), f_{A_0}(t_0)\big) &\leqslant H\big(f_{A_n}(t_n), f_{A_0}(t_n)\big) + H\big(f_{A_0}(t_n), f_{A_0}(t_0)\big) \\ &\leqslant H(A_n, A_0) + H\big(f_{A_0}(t_n), f_{A_0}(t_0)\big); \end{aligned}$$

let n tend to ∞; it follows from (III) and the hypothesis that we have

$$\lim_{n\to\infty} H\big(f_{A_n}(t_n), f_{A_0}(t_0)\big) = 0.$$

In the rest of this paper we assume that F is an (S)-convex structure on the metric space (X, d).

LEMMA 3. *For $0 \leqslant t \leqslant 1$, the mapping $x \to f_{\{x\}}(t)$ is nonexpansive from X into 2^X. Moreover, for all $A \in 2^X$ and $t \in [0, 1]$*

$$\gamma(A) \geqslant \gamma(f_A(t)).$$

PROOF. Due to (II) it is easy to see that the mapping $x \to f_{\{x\}}(t)$ is nonexpansive.

Next, if $r > \gamma(A)$, by Remark 2 there is a finite set G such that $H(G, A) \leqslant r$. By (V), $f_G(t)$ is compact, thus for any $c > 0$ there is a finite set J with $H(J, f_G(t)) < c$. By the triangle inequality and (II) we have

$$H(J, f_A(t)) \leqslant H(J, f_G(t)) + H(f_G(t), f_A(t)) < c + r.$$

By Remark 2, $\gamma(f_A(t)) < c + r$ for all $c > 0$ and $r > \gamma(A)$. Hence

$$\gamma(f_A(t)) \leqslant \gamma(A).$$

LEMMA 4. *For any $A \subset X$, let*

$$g(A) = \cup \{f_A^-(t) \mid 0 \leqslant t \leqslant 1\}.$$

Then $\gamma(g(A)) = \gamma(A)$.

PROOF. It is due to (5) in Lemma 1 and the definition of g, we may assume that $A \in 2^X$. Define a real-valued function h on $[0, 1] \times [0, 1]$ as follows:

$$h(s, t) = H(f_A(s), f_A(t)).$$

It follows from condition (III) that h is a continuous function on $[0, 1] \times [0, 1]$, and thus h is uniformly continuous. In particular, for $\varepsilon > 0$ there is a positive number δ such that

$$h(s, t) < \varepsilon \quad \text{for all } 0 \leqslant s, t \leqslant 1 \text{ and } |s - t| < \delta.$$

Hence for all $0 \leqslant s, t \leqslant 1$ and $|s - t| < \delta$ we have $f_A(s) \subset (f_A(t))_\varepsilon$. Take $P = \{t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1\}$ be a partition of $[0, 1]$ with

$$t_i - t_{i-1} < \delta \quad \text{for all } i = 1, 2, \ldots, n,$$

then $g(A) \subset \cup_{i=1}^n (f_A(t_i))_\varepsilon$.

From Lemmas 1 and 3, we have

$$\begin{aligned}\gamma(g(A)) &\leqslant \gamma\left(\bigcup_{i=1}^n (f_A(t_i))_\varepsilon\right)\\ &= \operatorname{Max}\{\gamma((f_A(t_i))_\varepsilon) \mid i = 1, 2, \ldots, n\}\\ &\leqslant \gamma(A) + \varepsilon.\end{aligned}$$

Therefore, $\gamma(g(A)) \leqslant \gamma(A)$. The fact that $A \subset g(A)$ implies

$$\gamma(g(A)) = \gamma(A).$$

REMARK 3. It follows from (IV) and the definition of g that for $A \subset B \subset X$ we have

$$f_A^-(t) \subset f_B^-(t) \quad \text{for all } t \in [0, 1]$$

and then

$$g(A) \subset g(B).$$

Now we are going to prove our theorem.

PROOF OF THE THEOREM. Let $x_0 \in X$. We denote O_{x_0} as the set

$$\cup \left\{(gT)^n x_0 \mid n = 0, 1, 2, \ldots\right\}$$

where $(gT)^0 x_0 = \{x_0\}$ and $(gT)^n x_0 = g(T((gT)^{n-1}x_0))$ for $n = 1, 2, \ldots$. Since $T(gT)^n x_0 \subset (gT)^{n+1}x_0$, we have

$$T(\overline{O}_{x_0}) \subset \overline{O}_{x_0} \quad \text{and} \quad \gamma(T(\overline{O}_{x_0})) \leqslant \gamma(\overline{O}_{x_0}).$$

Moreover, by (IV) and Remark 3 that

$$g(T(O_{x_0})) = g\left(\bigcup_{n=0}^{\infty} T(gT)^n x_0\right) \supset (gT)^{k+1}x_0$$

for all $k = 0, 1, 2 \ldots$; hence

$$\{x_0\} \cup gT(O_{x_0}) \supset \{x_0\} \cup \left(\bigcup_{n=0}^{\infty} (gT)^{n+1}x_0\right) = O_{x_0}$$

and thus

$$\gamma(\overline{O}_{x_0}) = \gamma(O_{x_0}) \leqslant \gamma(gT(\overline{O}_{x_0})) = \gamma(T(\overline{O}_{x_0})).$$

That is,

$$\gamma(\overline{O}_{x_0}) = \gamma(T(\overline{O}_{x_0})).$$

Since T is a condensing mapping, we have that $\overline{O}_{x_0}$ is a compact subset of X. Now we let $\{r_n\}$ be a strictly increasing sequence of positive numbers with $\lim_{n\to\infty} r_n = 1$ and let $K = \overline{O}_{x_0}$ and

$$T_n x = f_{Tx}(r_n) \quad \text{for all } x \in K.$$

For $y \in O_{x_0}$ we have $y \in (gT)^k x_0$ for some $k = 0, 1, 2, \ldots$. It follows from the definitions of T_n and O_{x_0} and Remark 3 that

$$T_n y = f_{Ty}(r_n) \subset g(Ty) \subset (gT)^{k+1}x_0 \subset O_{x_0}.$$

Then for any $x \in K$ there is a sequence $\{y_i\}$ in O_{x_0} which converges to x. By (II) we have

$$H(T_n y_i, T_n x) = H(f_{Ty_i}(r_n), f_{Tx}(r_n)) \leqslant H(Ty_i, Tx) \leqslant d(y_i, x);$$

hence $\lim_{i\to\infty} H(T_n y_i, T_n x) = 0$, or $T_n x \subset \overline{O}_{x_0} = K$, and thus T_n maps K into 2^K. Moreover, if x, y are in K we have

$$H(T_n x, T_n y) = H(f_{Tx}(r_n), f_{Ty}(r_n)) \leqslant \phi(r_n) H(Tx, Ty) \leqslant \phi(r_n) d(x, y);$$

that is, T_n is a contraction mapping of K into 2^K. It is due to a theorem of Nadler [3] that T_n has a fixed point x_n in K, or $x_n \in f_{Tx_n}(r_n)$. By the compactness of K there is a

convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, with y_0 as its limit. Hence, by Lemma 2, we have that

$$\lim_{k\to\infty} H\left(f_{Tx_{n_k}}(r_{n_k}), f_{Ty_0}(1)\right) = 0$$

and therefore $y_0 \in f_{Ty_0}(1) = Ty_0$.

It follows from our theorem and Remark 1 that we have the following:

COROLLARY. *Let X be a starlike convex, bounded, closed subset of a Banach space E. Then every nonexpansive condensing multi-valued mapping of X into 2^X has a fixed point.*

REMARK 4. If X is compact then every continuous mapping of X into 2^X is condensing, hence Dotson's theorem and its Corollary 2 are special cases of our theorem and corollary, respectively.

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