

TREES ARE CONTRACTIBLE

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ABSTRACT. Any hereditarily unicoherent, locally connected, compact connected Hausdorff space is contractible using an ordered continuum. An example is given of a hereditarily unicoherent, locally connected, first countable, compact connected Hausdorff space that does not admit the structure of a topological semigroup with zero and identity.

1. Definitions and examples. In this note we follow the terminology of [6], where a continuum is a nonempty compact connected Hausdorff space and an arc $[c, d]$ is a continuum with exactly two non-cutpoints c and d . Recall also that a space X is defined to be contractible to the point $p \in X$ if there exist both an arc $[c, d]$ and a continuous function $\phi: X \times [c, d] \rightarrow X$ such that for each $x \in X$, $\phi(x, c) = p$ and $\phi(x, d) = x$.

By [7, Theorem 9] the continuum X is a tree if and only if X is locally connected and hereditarily unicoherent. Given distinct points $p, q \in X$ there exists a unique arc $[p, q]$ contained in X with endpoints p and q . We use $[p, p]$ for $\{p\}$.

It is known that each subcontinuum of a tree is itself a tree. It follows from the main result of this paper that the continuum X is a tree if and only if X is locally connected and each subcontinuum of X is contractible.

We now define a first countable arc $B = [a, b]$. Let W denote the collection of all sequences $\{x_n\}$ of real numbers with the property that only finitely many x_n are nonzero and, for each n , $0 \leq x_n \leq 1$. Let $a \in W$ be the sequence with all entries equal to 0. Let b be the sequence with all entries equal to 1. Define " \leq " on $B = W \cup \{b\}$ by $\{x_n\} \leq \{y_n\}$ if either $x_n \leq y_n$ for all n or there exists a positive integer n such that $x_n < y_n$ and $x_m \leq y_m$ for all $m > n$.

Let (B, \leq) have the order topology. For each n , let $b_n \in B$ have the first n entries equal to 1 and the remaining entries equal to 0. Note that the arc $[a, b_n]$ is homeomorphic to $[0, 1]^n$, with the reverse dictionary order and the order topology.

PROPOSITION. *Let m be any positive integer. Let $f: [a, b_m] \rightarrow [a, b_{m+1}]$ be continuous. For $0 \leq t \leq 1$, let $A(t)$ denote the set of all $\{x_n\} \in B$ such that $x_{m+1} = t$ and $x_n = 0$, for all $n \geq m + 2$. For each t , $A(t)$ is homeomorphic to $A(0) = [a, b_m]$. Then for some t , $f([a, b_m])$ is contained in $A(t)$. Thus if $f(a) = a$, then $f([a, b_m])$ is contained in $[a, b_m]$.*

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PROOF. If $m = 1$, we note that the arc $[a, b_1]$ is separable and each $A(t)$ is a maximal separable subarc of $[a, b_2]$. We now use induction to prove the proposition when $m \geq 2$. We may assume that the following result has already been established: if the function $g: [a, b_{m-1}] \rightarrow [a, b_m]$ is continuous then, for some t , $g([a, b_{m-1}])$ is contained in $B(t)$, where $B(t)$ is the set of all $\{x_n\} \in B$ such that $x_m = t$ and $x_n = 0$, for all $n \geq m + 1$.

Define $H: [a, b_{m+1}] \rightarrow [a, b_m]$ by setting $H(\{x_n\}) = \{y_n\}$, where $y_n = x_{n+1}$, for $1 \leq n \leq m$, and $y_n = 0$, for all $n \geq m + 1$. Now H is continuous and for each $\{y_n\}$, $H^{-1}(\{y_n\})$ is a maximal separable subarc of $[a, b_{m+1}]$. We note that $H^{-1}(B(t)) = A(t)$, for $0 \leq t \leq 1$.

Let $h: [a, b_m] \rightarrow [a, b_{m-1}]$ be the restriction of H to $[a, b_m]$. Since the image of a separable space is itself separable, there exists a continuous function $g: [a, b_{m-1}] \rightarrow [a, b_m]$ such that $(g \circ h) = (H \circ f)$. By assumption, we conclude that $g([a, b_{m-1}])$ is contained in $B(t)$, for some $0 \leq t \leq 1$. But then $f([a, b_m])$ is contained in $H^{-1}(B(t)) = A(t)$.

Let $f: B \rightarrow B$ be continuous. By the Proposition, if $f(a) = a$ then $f([a, b_m])$ is contained in $[a, b_m]$, for all m . Another consequence of the Proposition is that $f(a) = b$ implies f is constant.

Now $B = [a, b]$ is a remarkable continuum which can be used to construct other spaces of interest. Let P be the first countable continuum obtained by identifying the non-cutpoints of B . The hyperspace of all subcontinua of P is not a contractible space, by a proof similar to that of [6, Theorem 5].

It is known [3, Theorem A] that each metric tree admits the structure of a topological semilattice with zero and identity. It follows that each metric tree is contractible (using a metric arc). Eberhart [1] gives an example of a nonmetrizable tree which does not admit the structure of a topological semigroup with zero and identity. We modify his work to produce a first countable example.

Let $S = \{1, 2, 3, 4, 5, 6\}$ have the discrete topology and let X be the space obtained from $B \times S$ by identifying:

- $(b, 1)$ with $(a, 2)$,
- $(b, 2)$ with $(a, 3)$ and $(a, 5)$,
- $(b, 3)$ with $(a, 4)$,
- $(b, 5)$ with $(b, 6)$.

Let $\psi: B \times S \rightarrow X$ be the identification map. Let $q_n = \psi(b, n)$, $1 \leq n \leq 5$, and let $q_0 = \psi(a, 1)$, $q_6 = \psi(a, 6)$. Then the continuum X is a first countable tree with exactly three non-cutpoints q_0 , q_4 , and q_6 . We claim that X is the desired example.

Suppose not, i.e. there is a continuous multiplication $\phi: X \times X \rightarrow X$ with zero $p \in X$ and identity $q \in X$. By [2, Exercise 17, p. 169], the identity element q must be a non-cutpoint of X . We sketch the proof for the case $q = q_4$ (the other two cases being handled in a similar manner).

Suppose $p = q_0$. Let $x = q_6$ and define the retraction map $r: X \rightarrow [p, x]$ by

$r(y) = y$, if $y \in [p, x]$, and $r(y) = q_2$, if $y \in [q_2, q]$. Define $F: [p, q] \rightarrow [p, x]$ by $F(z) = (r \circ \phi)(x, z)$, for each $z \in [p, q]$. Then $F(p) = p$ and $F(q) = x$. But the function F is continuous and $[q_0, q_1]$ is homeomorphic to B . By the Proposition, $F([q_0, q_1])$ is contained in $[q_0, q_1]$. At best we have $F(q_1) = q_1$ and then (by the Proposition) $F([q_1, q_2])$ is contained in $[q_0, q_2]$. At best we have $F(q_2) = q_2$ and then (by the Proposition) $F([q_2, q_3])$ is contained in $[q_0, q_5]$. At best we have $F(q_3) = q_5$. But $[q_2, q_6]$ is homeomorphic to the union of two copies of B , meeting at the point b in each arc. It is then a consequence of the Proposition that $F([q_3, q_4])$ equals $\{q_5\}$. We conclude that we always have $F([q_0, q_4])$ contained in $[q_0, q_5]$ and thus $F(q) = F(q_4)$ is not equal to $x = q_6$. A contradiction has been reached. Essentially the same proof works for any point $p \in [q_0, q_4]$.

Suppose $p \in [q_2, q_6]$. Then by choosing a retraction map $r: X \rightarrow [p, x]$, where $x = q_0$, and defining $F: [p, q] \rightarrow [p, x]$ by $F(z) = (r \circ \phi)(x, z)$, we can again use the Proposition to reach the contradictory conclusion: $F(q)$ is not equal to x . This ends the proof for the case $q = q_4$.

We note that the preceding proof also shows that there is no subarc $[p, q]$ of X such that X is contractible to the point $p \in X$, using the arc $[p, q]$.

2. Proof of main result. Let T be any tree containing more than one point and let $C(T)$ be the hyperspace of all subcontinua of T with the finite topology [5]. We shall prove that, for each $t \in T$, the space T is contractible to the point t using some arc $[c, d]$ contained in the hyperspace $C(T)$. We note that hyperspaces are arcwise connected [4]. For each subcontinuum $A \in C(T)$ there is a unique continuous retraction $h_A: T \rightarrow A$ such that, for each $x \in A$, $\{t \in T: h_A(t) = x\}$ is a continuum. Define $h: T \times C(T) \rightarrow T$ by, for each $t \in T$ and each $A \in C(T)$, $h(t, A) = h_A(t)$. We note that for all $s, t \in T$ we have $h(s, \{t\}) = t$ and $h(s, T) = s$. Given the fact that there is an arc in the hyperspace joining $\{t\}$ and T , we have only to prove that the function h is continuous.

Consider a net $\{(t_n, A_n): n \in \Gamma\}$ converging to (t, A) in $T \times C(T)$, such that the net $\{h(t_n, A_n): n \in \Gamma\}$ converges to $y \in T$. We show that $y = h(t, A)$.

Let U be any open set containing A . Then there exists an element m such that, for all $n \geq m$, $A_n \in U$. Thus $h(t_n, A_n) \in U$. Therefore y is in the closure of U . We conclude that $y \in A$.

Let $q \in A$, $q \neq h(t, A)$. Choose a cutpoint p in the arc $[q, h(t, A)]$. Let V be the component of $(X - \{p\})$ containing $h(t, A)$. Let U be the component of $(X - \{h(t, A)\})$ containing p . Then U, V are open sets and $t \in V$. Let $\langle U, V \rangle$ denote the nbhd of A in the hyperspace consisting of all subcontinua $K \subset (U \cup V)$ having nonempty intersection with both U and V . Then there exists an element m such that, for all $n \geq m$, we have $t_n \in V$ and $A_n \in \langle U, V \rangle$. Thus there exists a point $x_n \in (A_n \cap V)$. But then $h(t_n, A_n)$ is contained in $[t_n, x_n]$ which is contained in V . Therefore y is in the closure of V and so $y \neq q$. We conclude that $y = h(t, A)$. This completes the proof.

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