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ON A CHARACTERIZATION OF INVARIANT SUBSPACE LATTICES OF WEIGHTED SHIFTS

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ABSTRACT. The paper concerns itself with the characterization of invariant subspace lattices of weighted shift operators on the Hilbert space l^2 with suitable conditions on their weights. This characterization is also extended to the case of Banach spaces l^p, $1 < p < \infty$.

1. Introduction. Let l^2 be the Hilbert space of all square-summable complex sequences $x = \{x_0, x_1, x_2, \ldots\}$ with the norm

$$\|x\| = \left(\sum_{m=0}^{\infty} |x_m|^2 \right)^{1/2}.$$

If $\{w_m\}_{m=0}^{\infty}$ is a bounded sequence of nonzero complex numbers, then the operator T, defined by

$$T\{x_0, x_1, x_2, \ldots\} = \{0, w_0x_0, w_1x_1, w_2x_2, \ldots\},$$

is called a unilateral (forward) weighted shift on l^2 with the weight sequence $\{w_m\}_{m=0}^{\infty}$. We may, and shall, assume without any loss of generality that the weights w_m are positive real numbers [**2**, Problem 2]. By an invariant subspace M of T we shall mean a closed linear manifold of l^2 such that $TM \subset M$. We shall denote by $\operatorname{Lat} T$ the lattice of invariant subspaces of T. Various authors have characterized $\operatorname{Lat} T$ under suitable conditions on the weight sequence $\{w_m\}_{m=0}^{\infty}$. The following characterization is due to Nikolskiĭ [**3**]:

If the weight sequence $\{w_m\}_{m=0}^{\infty}$ is monotonically decreasing to zero and belongs to l^2, then

$$\operatorname{Lat} T = \left\{\{0\}, M_1, M_2, \ldots, M_k, \ldots, l^2\right\}, \tag{1}$$

where

$$M_k = \left\{x \in l^2 \colon x_m = 0, m < k\right\}.$$

The particular case of this result in which $w_m = 2^{-m}$ was originally obtained by Donoghue [**1**]; see also [**5**, p. 66]. The central theme of this paper is to exhibit that Nikolskiĭ's theorem holds under more general conditions on the weight sequence

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$\{w_m\}_{m=0}^{\infty}$. We shall say that the sequence $\{w_m\}_{m=0}^{\infty}$ is of bounded variation if

$$\sum_{m=0}^{\infty} |w_m - w_{m+1}| < \infty.$$

It is easy to see that if $\{w_m\}_{m=0}^{\infty}$ is monotonically decreasing, then it is of bounded variation, but the converse is not true.

2. We now prove

THEOREM 1. *If the weight sequence $\{w_m\}_{m=0}^{\infty}$ is of bounded variation and*

$$\delta = \sup_{m \geqslant 2, n} \sum_{k=0}^{\infty} \left(\frac{w_{k+m} \cdots w_{k+n}}{w_m w_{m+1} \cdots w_n} \right)^2 < \infty, \tag{2}$$

then Lat T *is given by* (1).

PROOF. Let $\{e_m\}_{m=0}^{\infty}$ be the standard orthonormal basis of l^2 and let M be an invariant subspace of T. Firstly, we proceed to show that if a vector $x = \{x_m\}_{m=0}^{\infty}$, with $x_0 \neq 0$, is in M, then $M = l^2$. As

$$T^n x = \Big\{ \underbrace{0, 0, \ldots, 0}_{n}, x_0 w_0 w_1 \cdots w_{n-1}, x_1 w_1 w_2 \cdots w_n, \ldots \Big\},$$

it follows that

$$\begin{aligned}
\left\| \frac{T^n x}{x_0 w_0 w_1 \cdots w_{n-1}} - e_n \right\|^2 &= \sum_{m=0}^{\infty} \left(\frac{w_{m+1} \cdots w_{m+n}}{w_0 w_1 \cdots w_{n-1}} \right)^2 \left| \frac{x_{m+1}}{x_0} \right|^2 \\
&= \frac{w_n^2}{w_0^2 |x_0|^2} \sum_{m=0}^{\infty} \left(\frac{w_{m+1} \cdots w_{m+n}}{w_1 \cdots w_n} \right)^2 |x_{m+1}|^2 \\
&\leqslant \frac{w_n^2 \|x\|^2}{w_0^2 |x_0|^2} \sum_{m=0}^{\infty} \left(\frac{w_{m+1} \cdots w_{m+n}}{w_1 \cdots w_n} \right)^2 \\
&= \frac{w_n^2 \|x\|^2}{w_0^2 w_1^2 |x_0|^2} \sum_{m=0}^{\infty} \left(\frac{w_{m+2} \cdots w_{m+n}}{w_2 \cdots w_n} \right)^2 w_{m+1}^2 \\
&= \frac{w_n^2 \|x\|^2}{w_0^2 w_1^2 |x_0|^2} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left(\frac{w_{k+2} \cdots w_{k+n}}{w_2 \cdots w_n} \right)^2 (w_{m+1}^2 - w_{m+2}^2) \\
&\qquad \text{(by Abel's transformation [7])} \\
&\leqslant \frac{\delta w_n^2 \|x\|^2}{w_0^2 w_1^2 |x_0|^2} \sum_{m=0}^{\infty} (w_{m+1}^2 - w_{m+2}^2) \qquad \text{(by (2))} \\
&\leqslant \frac{\delta w_n^2 \|x\|^2}{w_0^2 w_1^2 |x_0|^2} \sum_{m=0}^{\infty} |w_{m+1} - w_{m+2}| (w_{m+1} + w_{m+2}) \\
&\leqslant \frac{2\delta \mu w_n^2 \|x\|^2}{w_0^2 w_1^2 |x_0|^2} \sum_{m=0}^{\infty} |w_{m+1} - w_{m+2}| \leqslant C w_n^2,
\end{aligned}$$

where $\mu = \sup_m\{w_m\}$ and C is a constant. Since $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis in l^2 and $\Sigma_{n=0}^{\infty} w_n^2 < \infty$, it follows by the Paley-Wiener theorem [**6**, p. 208] that the system

$$\left\{\frac{T^n x}{x_0 w_0 w_1 \cdots w_{n-1}}\right\}_{n=0}^{\infty}$$

is a Riesz basis in l^2, whence we conclude that $M = l^2$. Again, if $x_0 = 0$ and k is the least natural number such that $x_k \neq 0$, then we can similarly show that

$$\bigvee_{n=0}^{\infty} \{T^n x\} = M_k.$$

Thus we have shown that every cyclic subspace of T is an M_k. Now our theorem follows by observing that the span of any number of M_k's is again an M_k.

Our next theorem, which we state without proof, shows that the condition of bounded variation can be dispensed with, provided that condition (2) is replaced by a more stringent condition; even so, our theorem generalizes Nikolskiĭ's result.

THEOREM 2. *If the weight sequence* $\{w_m\}_{m=0}^{\infty}$ *satisfies the condition*

$$\delta = \sup_{m \geqslant 1, n} \sum_{k=0}^{\infty} \left(\frac{w_{k+m} \cdots w_{k+n}}{w_m w_{m+1} \cdots w_n}\right)^2 < \infty, \tag{3}$$

then Lat T *is given by* (1).

Now we extend Theorem 2 for the l^p spaces, $1 < p < \infty$, which, in turn, generalizes Nikolskiĭ's main result [**3**, Theorem 2]. We shall denote by q the Hölder conjugate of p, i.e., the number determined by $1/p + 1/q = 1$.

THEOREM 3. *Let T be a unilateral (forward) weighted shift on l^p with weight sequence* $\{w_m\}_{m=0}^{\infty}$ *and let*

$$\delta = \sup_{m \geqslant 1, n} \sum_{k=0}^{\infty} \left(\frac{w_{k+m} \cdots w_{k+n}}{w_m w_{m+1} \cdots w_n}\right)^q < \infty. \tag{4}$$

Then Lat T *is given by*

$$\operatorname{Lat} T = \{\{0\}, M_1, M_2, \dots, M_k, \dots, l^p\},$$

where

$$M_k = \{x \in l^p : x_m = 0, m < k\}.$$

We shall only sketch the proof of this theorem. Let M be any subspace of l^p invariant under T. If a vector $x = \{x_m\}_{m=0}^{\infty}$, $x_0 \neq 0$, is in M, we intend to show that $M = l^p$. Let $y = \{y_m\}_{m=0}^{\infty}$ be any element in l^q such that $y(T^n x) = 0$, $n = 0, 1, 2, \dots$. It will suffice to show that $y = 0$. We have, for $x_0 = 1$,

$$y_n = \frac{-1}{w_0 w_1 \cdots w_{n-1}} \sum_{m=0}^{\infty} w_{m+1} w_{m+2} \cdots w_{m+n} x_{m+1} y_{m+n+1},$$

and hence, using Hölder's inequality, Abel's transformation and condition (4) in succession, we obtain

$$
\begin{aligned}
|y_n| &\leqslant \frac{w_n}{w_0}\left(\sum_{m=0}^{\infty}|x_{m+1}|^p\right)^{1/p}\left(\sum_{m=0}^{\infty}\left(\frac{w_{m+1}\cdots w_{m+n}}{w_1w_2\cdots w_n}\right)^q|y_{m+n+1}|^q\right)^{1/q}\\
&\leqslant \frac{w_n}{w_0}\|x\|\left(\sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\frac{w_{k+1}\cdots w_{k+n}}{w_1w_2\cdots w_n}\right)^q(|y_{m+n+1}|^q-|y_{m+n+2}|^q)\right)^{1/q}\\
&\leqslant \frac{w_n}{w_0}\|x\|\delta^{1/q}\left(\sum_{m=0}^{\infty}(|y_{m+n+1}|^q-|y_{m+n+2}|^q)\right)^{1/q}\\
&\leqslant 2\frac{w_n}{w_0}\delta^{1/q}\|x\|\,\|y\|.
\end{aligned}
$$

This inequality is the main step in the proof of Nikolskiĭ's theorem [3]. By following exactly the line of his argument, we can show that $y = 0$, and we are done.

If $x = \{x_m\}_{m=0}^{\infty}$ is a vector in M with $x_0 = 0$, let k be the least positive integer such that $x_k \neq 0$. By repeating the argument used above, we obtain $M = M_k$. This completes the proof.

3. We now consider the Hilbert space $l^2(\mathbf{C}^k)$, $k \geqslant 1$, of norm-square-summable sequences of vectors of the k-dimensional unitary spaces $\mathbf{C}^k$. Thus $l^2(\mathbf{C}^k)$ consists of sequences

$$x = \{x_m\}_{m=0}^{\infty}, \qquad x_m \in \mathbf{C}^k,$$

such that

$$\sum_{m=0}^{\infty}\|x_m\|_*^2 < \infty,$$

where $\|x_m\|_*$ is the norm of x_m in $\mathbf{C}^k$, and

$$\|x\| = \left(\sum_{m=0}^{\infty}\|x_m\|_*^2\right)^{1/2}.$$

We shall say that a nonempty subset S of $l^2(\mathbf{C}^k)$ is a cyclic set of an operator T on $l^2(\mathbf{C}^k)$ if

$$\bigvee_{n=0}^{\infty}\{T^n x\colon x \in S\} = l^2(\mathbf{C}^k).$$

The following theorem generalises a result due to Nikolskiĭ [4, Lemma 1].

THEOREM 4. *Let T be a unilateral weighted shift on $l^2(\mathbf{C}^k)$ with weight sequence $\{w_m\}_{m=0}^{\infty}$ such that $\{w_m\}_{m=0}^{\infty}$ is of bounded variation and*

$$\delta = \sup_n \sum_{k=0}^{\infty}\left(\frac{w_{k+2}\cdots w_{k+n}}{w_2\cdots w_n}\right)^2 < \infty. \tag{5}$$

Then any set of k-vectors in $l^2(\mathbf{C}^k)$, such that their first coordinates form a basis of $\mathbf{C}^k$, is a cyclic set of T.

PROOF. Let $x^{(i)} = \{x_m^{(i)}\}_{m=0}^{\infty}$, $i = 1, 2, \ldots, k$, be k elements of $l^2(\mathbf{C}^k)$ such that $\{x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}\}$ is a basis in $\mathbf{C}^k$. We assume without any loss of generality that $\{x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}\}$ is an orthonormal basis in $\mathbf{C}^k$. Then

$$T^n x^{(i)} = \Big\{ \underbrace{0, 0, \ldots, 0}_{n}, w_{n-1} \cdots w_1 w_0 x_0^{(i)}, w_n \cdots w_2 w_1 x_1^{(i)}, \ldots \Big\}.$$

For each $z \in \mathbf{C}^k$, let $e_n(z)$ denote the element of $l^2(\mathbf{C}^k)$ having z in the nth place and 0 elsewhere. Now observing that

$$\|x_{m+1}^{(i)}\|_* \leqslant \|x^{(i)}\|$$

and, as in the proof of Theorem 1,

$$\sum_{m=0}^{\infty} \left(\frac{w_{m+1} \cdots w_{m+n}}{w_0 w_1 \cdots w_{n-1}} \right)^2 \leqslant C w_n^2,$$

we have

$$\begin{aligned} \left\| \frac{T^n x^{(i)}}{w_0 w_1 \cdots w_{n-1}} - e_n\big(x_0^{(i)}\big) \right\|^2 &= \sum_{m=0}^{\infty} \left(\frac{w_{m+1} \cdots w_{m+n}}{w_0 w_1 \cdots w_{n-1}} \right)^2 \|x_{m+1}^{(i)}\|_*^2 \\ &\leqslant C w_n^2 \|x^{(i)}\|^2. \end{aligned}$$

Since

$$\big\{ e_n\big(x_0^{(i)}\big) \big\}_{n \geqslant 0,\ 1 \leqslant i \leqslant k}$$

is an orthonormal basis in $l^2(\mathbf{C}^k)$ and

$$\sum_{\substack{n \geqslant 0 \\ 1 \leqslant i \leqslant k}} w_n^2 \|x^{(i)}\|^2 < \infty,$$

it follows that the system

$$\left\{ \frac{T^n x^{(i)}}{w_0 w_1 \cdots w_{n-1}} \right\}_{n \geqslant 0,\ 1 \leqslant i \leqslant k}$$

is a Riesz basis in $l^2(\mathbf{C}^k)$, whence we conclude that $\{x^{(i)}\}_{i=1}^k$ is a cyclic set of T.

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