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OSCILLATION THEOREMS FOR A SECOND ORDER SUBLINEAR ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT. New oscillation criteria are given for the differential equation

$$u'' + a(t)\,|u|^{\alpha}\operatorname{sgn} u = 0, \qquad 0 < \alpha < 1,$$

where $a(t)$ is allowed to take on negative values for arbitrarily large t.

We consider the sublinear differential equation

$$u'' + a(t)\,|u|^{\alpha}\operatorname{sgn} u = 0, \qquad 0 < \alpha < 1, \tag{1}$$

where $a(t)$ is continuous on $[t_0, \infty)$, $t_0 > 0$. We restrict our attention to proper solutions of (1), that is, those solutions $u(t)$ which exist on some ray $[T_u, \infty) \subset [t_0, \infty)$ and satisfy $\sup\{|u(t)|: t \geqslant T\} > 0$ for any $T \geqslant T_u$. A proper solution is called oscillatory if it has arbitrary large zeros; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all of its proper solutions are oscillatory.

Belohorec [1] has shown that, in case $a(t)$ is nonnegative, equation (1) is oscillatory if and only if

$$\int_{t_0}^{\infty} t^{\alpha} a(t)\,dt = \infty. \tag{2}$$

Of particular interest, therefore, is the problem of finding criteria for the oscillation of equation (1) when $a(t)$ is allowed to take on negative values for arbitrarily large t. This problem has been studied by various authors including Belohorec [2], Butler [3, 4], Grammatikopoulos [5], Kamenev [6] and Wong [7]. We refer in particular to the following oscillation criteria for (1) due, respectively, to Belohorec [2] and Kamenev [6]:

$$\int_{t_0}^{\infty} t^{\beta} a(t)\,dt = \infty \quad \text{for some } \beta \in [0, \alpha]; \tag{3}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} a(\tau)\,d\tau\,ds = \infty. \tag{4}$$

The purpose of this paper is to proceed further in this direction to present new oscillation theorems which unify and considerably improve the above-mentioned results of Belohorec and Kamenev.

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Our main results are as follows:

THEOREM 1. *Equation* (1) *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} \tau^{\beta} a(\tau)\, d\tau\, ds = \infty \quad \text{for some } \beta \in [0, \alpha]. \tag{5}$$

THEOREM 2. *Let* $\beta \in [0, \alpha)$. *Equation* (1) *is oscillatory if there exists a continuous function* $f(t)$ *on* $[t_0, \infty)$ *such that*

$$\limsup_{T\to\infty} \frac{1}{T}\int_{t}^{T}\int_{t}^{s} \tau^{\beta} a(\tau)\, d\tau\, ds \geqslant f(t), \tag{6}$$

for $t \in [t_0, \infty)$ *and*

$$\int_{t_0}^{\infty} \frac{f_+^2(t)}{t}\, dt = \infty, \tag{7}$$

where $f_+(t) = \max\{f(t), 0\}$.

PROOF OF THEOREMS 1 AND 2. Suppose to the contrary that there exists a nonoscillatory solution $u(t)$ of (1). Without loss of generality we may suppose that $u(t) > 0$ on $[t_1, \infty)$, $t_1 > t_0$. Put

$$v(t) = \frac{1}{1-\alpha} t^{\beta} u^{1-\alpha}(t).$$

It is easy to verify that

$$v''(t) = -\frac{\alpha}{1-\alpha}\frac{1}{v(t)}\left(v'(t) - \frac{\beta v(t)}{\alpha t}\right)^2 - \frac{\beta(\alpha-\beta)}{\alpha}\frac{v(t)}{t^2} - t^{\beta} a(t). \tag{8}$$

Integrating (8) twice over $[t, T]$, $t \geqslant t_1$, we have

$$\begin{aligned} v(t) - v(T) + v'(t)(T-t) = {} & \frac{\alpha}{1-\alpha}\int_t^T\int_t^s \frac{1}{v(\tau)}\left(v'(\tau) - \frac{\beta v(\tau)}{\alpha\tau}\right)^2 d\tau\, ds \\ & + \frac{\beta(\alpha-\beta)}{\alpha}\int_t^T\int_t^s \frac{v(\tau)}{\tau^2}\, d\tau\, ds + \int_t^T\int_t^s \tau^{\beta} a(\tau)\, d\tau\, ds. \end{aligned} \tag{9}$$

Suppose (5) holds for some $\beta \in [0, \alpha]$. Divide (9) by T and take the upper limit as $T \to \infty$. Using (5), we then see that

$$v'(t) \geqslant \limsup_{T\to\infty} \frac{1}{T}\int_t^T\int_t^s \tau^{\beta} a(\tau)\, d\tau\, ds = \infty$$

for all $t \geqslant t_1$. This contradiction proves Theorem 1.

Let $\beta \in [0, \alpha)$ and suppose there is a function $f(t)$ satisfying (6) and (7). Dividing (9) by T, taking the upper limit as $T \to \infty$ and using (6), we obtain

$$\begin{aligned} & \liminf_{T\to\infty} \frac{v(T)}{T} + \frac{\alpha}{1-\alpha}\int_t^{\infty} \frac{1}{v(s)}\left(v'(s) - \frac{\beta v(s)}{\alpha s}\right)^2 ds \\ & \qquad + \frac{\beta(\alpha-\beta)}{\alpha}\int_t^{\infty}\frac{v(s)}{s^2}\, ds + f(t) \leqslant v'(t), \qquad t \geqslant t_1, \end{aligned} \tag{10}$$

which shows that

$$\liminf_{T\to\infty}\frac{v(T)}{T}<\infty,\quad \int_t^\infty \frac{1}{v(s)}\left(v'(s)-\frac{\beta v(s)}{\alpha s}\right)^2 ds<\infty,\tag{11}$$
$$\int_t^\infty \frac{v(s)}{s^2}ds<\infty$$

for $t \geqslant t_1$. On the other hand, integrating the equality

$$\frac{2\beta}{\alpha}\left(\frac{v(t)}{t}\right)'+\frac{1}{v(t)}\left(v'(t)-\frac{\beta v(t)}{\alpha t}\right)^2=\frac{v'^2(t)}{v(t)}+\frac{\beta^2-2\alpha\beta}{\alpha^2}\frac{v(t)}{t^2}$$

we have

$$\frac{2\beta}{\alpha}\left(\frac{v(T)}{T}-\frac{v(t)}{t}\right)+\int_t^T\frac{1}{v(s)}\left(v'(s)-\frac{\beta v(s)}{\alpha s}\right)^2 ds\tag{12}$$
$$=\int_t^T\frac{v'^2(s)}{v(s)}ds+\frac{\beta^2-2\alpha\beta}{\alpha^2}\int_t^T\frac{v(s)}{s^2}ds.$$

Taking the lower limit as $T\to\infty$ in (12), we conclude with the aid of (11) that

$$\int_t^\infty\frac{v'^2(s)}{v(s)}ds<\infty,\qquad t\geqslant t_1.\tag{13}$$

By Schwarz's inequality we have.

$$t\int_{t_1}^\infty\frac{v'^2(s)}{v(s)}ds\geqslant\left(\int_{t_1}^t\frac{v'(s)}{v^{1/2}(s)}ds\right)^2=4\left(v^{1/2}(t)-v^{1/2}(t_1)\right)^2,$$

which, together with (13), implies that

$$v(t)\leqslant kt,\quad t\geqslant t_2,\tag{14}$$

for some constants $k>0$ and $t_2>t_1$. Since $\max\{v'(t),0\}\geqslant f_+(t)\geqslant 0$ by (10), it follows from (13) and (14) that

$$\frac{1}{k}\int_{t_2}^\infty\frac{f_+^2(t)}{t}dt\leqslant\int_{t_2}^\infty\frac{v'^2(t)}{v(t)}dt<\infty$$

which contradicts (7). This completes the proof of Theorem 2.

EXAMPLE 1. Consider the equation

$$u''+(t^\gamma\sin t)\,|u|^\alpha\operatorname{sgn}u=0,\qquad 0<\alpha<1,\tag{15}$$

for $t>0$, where γ is a constant. If $\beta+\gamma>1$, then

$$\limsup_{T\to\infty}\frac{1}{T}\int_t^T\int_t^s\tau^{\beta+\gamma}\sin\tau\,d\tau\,ds=\infty,$$

so that, by Theorem 1, (15) is oscillatory when $\gamma>1-\alpha$. If $0<\beta+\gamma\leqslant 1$, then we have

$$\limsup_{T\to\infty}\frac{1}{T}\int_t^T\int_t^s\tau^{\beta+\gamma}\sin\tau\,d\tau\,ds\geqslant t^\delta\cos t-\delta t^{\delta-1}\sin t$$
$$-\delta(\delta-1)t^{\delta-2}\cos t-\delta(\delta-1)(\delta-2)\int_t^\infty s^{\delta-3}\cos s\,ds,$$

where $\delta = \beta + \gamma$. Put $f(t) = t^{\delta}\cos t - 2\delta$. Then (6) holds for $t \geqslant t_1$, provided t_1 is sufficiently large. Since $\delta > 0$, there is an integer $N > 0$ such that $2N\pi - (\pi/4) > t_1$ and

$$f(t) \geqslant 2^{-1/2} \quad \text{on } [2n\pi - \pi/4, 2n\pi + \pi/4],$$

for all $n \geqslant N$. It follows that

$$\int_{t_1}^{\infty} \frac{f_+^2(t)}{t}\,dt \geqslant \sum_{n=N}^{\infty} \int_{2n\pi-(\pi/4)}^{2n\pi+(\pi/4)} \frac{dt}{2t} = \frac{1}{2}\sum_{n=N}^{\infty} \log\left(1 + \frac{2}{8n-1}\right) = \infty,$$

and so (7) is satisfied. Applying Theorem 2, we see that (15) is oscillatory if $-\alpha < \gamma \leqslant 1 - \alpha$. We conclude therefore that equation (15) is oscillatory if $\gamma > -\alpha$. (Butler [4] conjectures that (15) is oscillatory if and only if $\gamma \geqslant -\alpha$.)

We note that Belohorec's criterion (3) does not apply to (15) and that Kamenev's criterion (4) assures the oscillation of (15) only for $\gamma > 1$.

REMARK. In Theorem 2 the assumption that $\beta < \alpha$ is essential. In fact, consider the equation (1) in which $a(t) = t^{-\alpha-1}(\log t)^{-3/2}$ for $t > 1$. As easily verified,

$$\limsup_{T\to\infty} \frac{1}{T}\int_t^T \int_t^s \tau^{\alpha} a(\tau)\,d\tau\,ds = 2(\log t)^{-1/2}, \qquad t > 1,$$

so that conditions (6) and (7) are satisfied with $\beta = \alpha$ and $f(t) = 2(\log t)^{-1/2}$. However, since $a(t) > 0$ and $\int^{\infty} t^{\alpha} a(t)\,dt < \infty$, the equation in question is not oscillatory according to Belohorec's theorem [1].

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