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SPLITTING UNIVERSAL BUNDLES OVER FLAG MANIFOLDS

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ABSTRACT. Let $\mathbf{F}$ be one of the fields $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and correspondingly let $\mathbf{F}G$ be O, U, or Sp, i.e. the orthogonal, unitary, or symplectic group. Over the flag manifold $\mathbf{F}G(n_1 + \cdots + n_k)/\mathbf{F}G(n_1) \times \cdots \times \mathbf{F}G(n_k)$ one has vector bundles γ_i over F of dimension n_i, $1 \leqslant i \leqslant k$. This paper determines all cases in which γ_i decomposes nontrivially as a Whitney sum.

1. Introduction. Let $\mathbf{F}$ be one of the fields $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, and correspondingly let $\mathbf{F}G$ be O, U, or Sp, i.e. the orthogonal, unitary, or symplectic group. If $n = n_1 + n_2 + \cdots + n_k$, $1 \leqslant n_i$, let

$$\mathbf{F}M(n_1, n_2, \ldots, n_k) = \mathbf{F}G(n)/\mathbf{F}G(n_1) \times \mathbf{F}G(n_2) \times \cdots \times \mathbf{F}G(n_k)$$

be the manifold of flags, consisting of k mutually orthogonal $\mathbf{F}$-subspaces of $\mathbf{F}^n$ with the ith subspace having dimension n_i. Over $\mathbf{F}M(n_1, n_2, \ldots, n_k)$, one has vector bundles γ_i, $1 \leqslant i \leqslant k$, over $\mathbf{F}$ with the fiber dimension of γ_i being n_i, with the total space consisting of pairs (V, x), where $V = (V_1^{n_1}, V_2^{n_2}, \ldots, V_k^{n_k})$ is a flag and $x \in V_i^{n_i}$ is in the ith subspace.

The purpose of this note is to determine all cases in which it is possible to have decompositions of the bundles γ_i into nontrivial Whitney sums of $\mathbf{F}$-subbundles. Equivalently, let $n_i = n_1^i + n_2^i + \cdots + n_{j_i}^i$, $1 \leqslant n_j^i$, $1 \leqslant j_i$, be partitions of the n_i. One then has a fibering

$$\pi\colon \mathbf{F}M\left(n_1^1, \ldots, n_{j_1}^1, \ldots, n_1^k, \ldots, n_{j_k}^k\right) \to \mathbf{F}M(n_1, n_2, \ldots, n_k),$$

and this note will determine all cases in which π has a section.

This problem is suggested by work of Glover and Homer **[1, 2]**, and generalizes the solution for projective spaces (the case $\mathbf{F}M(1, n_2)$) in **[3]** and for Grassmannians (the case $\mathbf{F}M(p, q)$, $2 \leqslant p \leqslant q$) in **[5]**. In essence, this note reduces the general problem to those special cases, or more precisely to the solutions given for those cases. The easiest case is $\mathbf{F} = \mathbf{H}$, which is in fact completely trivial, while the case $\mathbf{F} = \mathbf{R}$ is the most difficult and requires a few new thoughts. The paper will be organized by taking these cases in the order of increasing difficulty.

In the final section, it will be shown that there are no real splittings of the universal bundles over quaternionic Grassmannians $\mathbf{H}M(p, q)$, $2 \leqslant p \leqslant q$. This case was not covered in **[5]**, and serves to complete those results.

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2. Easy results: the case $\mathbf{F} = \mathbf{H}$.

LEMMA 1. *Let $1 \leqslant n_i$, $1 \leqslant i \leqslant k$, with $2 \leqslant k$. Let $n_i = n_1^i + n_2^i + \cdots + n_{j_i}^i$, with $1 \leqslant n_j^i$, $1 \leqslant j_i$, and suppose that $j_k > 1$; i.e. one has a genuine partition of n_k. If the fibering*

$$\pi\colon \mathbf{F}M\left(n_1^1,\ldots,n_{j_1}^1,\ldots,n_1^k,\ldots,n_{j_k}^k\right) \to \mathbf{F}M(n_1, n_2,\ldots,n_k)$$

has a section, then the fibering

$$\pi'\colon \mathbf{F}M\left(\underbrace{1,\ldots,1}_{r}, n_1^k,\ldots,n_{j_k}^k\right) \to \mathbf{F}M\left(\underbrace{1,\ldots,1}_{r}, n_k\right)$$

also has a section, provided $r \leqslant n_1 + n_2 + \cdots + n_{k-1}$.

PROOF. If π has a section, γ_k splits over $\mathbf{F}M(n_1,\ldots,n_k)$ and hence the pullback over $\mathbf{F}M(1,\ldots,1, n_k)$ (r ones) admits the same decomposition. □

PROPOSITION 1. *In the quaternionic case* ($\mathbf{F} = \mathbf{H}$), *none of the genuine fiberings π has a section.*

PROOF. If $\pi\colon \mathbf{H}M(n_1^1,\ldots,n_{j_k}^k) \to \mathbf{H}M(n_1,\ldots,n_k)$, with $2 \leqslant k, j_k > 1$, has a section, then by the lemma, so does

$$\pi'\colon \mathbf{H}M\left(1, n_1^k,\ldots,n_{j_k}^k\right) \to \mathbf{H}M(1, n_k) = \mathbf{H}P^{n_k}.$$

However, the bundle γ_n over $\mathbf{H}P^n$ never admits a proper Whitney sum decomposition over $\mathbf{H}$ (or even over $\mathbf{C}$) by [3]. □

3. Semisimple results: the case $\mathbf{F} = \mathbf{C}$.

LEMMA 2. *The fibering*

$$\pi\colon \mathbf{C}M\left(1, 1, m_1,\ldots,m_j\right) \to \mathbf{C}M(1, 1, m)$$

with $j > 1$ has no section.

PROOF. Suppose there was a section. Over $\mathbf{C}M(1, 1, m)$, one has two line bundles γ_1 and γ_2 and an m-plane bundle γ_3, with the bundle γ_3 having a proper Whitney sum decomposition $\gamma_3 = \xi^p \oplus \eta^q$, $p + q = m$, $p \leqslant q$. By Lemma 1, the same decomposition exists over $\mathbf{C}M(1, m)$. Applying the results of [3], m is odd and $p = 1$, $q = m - 1$. Further, over $\mathbf{C}M(1, m)$, the line bundle ξ is the complex conjugate of the line bundle γ_1.

Thus, over $\mathbf{C}M(1, 1, 2s + 1)$, the only possible decomposition is $\gamma_3 \cong \xi^1 \oplus \eta^{2s}$. Let $c_1(\gamma_1) = x$ and $c_1(\gamma_2) = y$. Using the two different inclusions of $\mathbf{C}M(1, 2s + 1)$ in $\mathbf{C}M(1, 1, 2s + 1)$, it is immediate that $c_1(\xi) = -(x + y)$. Now examining the arguments in [5], the argument that γ^p over $\mathbf{C}M(2, p)$ cannot split was actually done by calculation in $\mathbf{C}P(\gamma^2) = \mathbf{C}M(1, 1, p)$. Using Stiefel-Whitney and Pontrjagin class calculations, it was shown that the pullback of γ^p to $\mathbf{C}P(\gamma^2)$ could not contain a line bundle with Chern class $-(x + y)$. □

PROPOSITION 2. *In the complex case* ($\mathbf{F} = \mathbf{C}$), *the only genuine fiberings π having sections are the fiberings*

$$\pi\colon \mathbf{C}M(1,1,2s) \to \mathbf{C}M(1,2s+1).$$

PROOF. If $\pi\colon \mathbf{C}M(n_1^1,\ldots,n_{j_k}^k) \to \mathbf{C}M(n_1,\ldots,n_k)$ has a section, then Lemma 1 gives a section of π' for $r \leq n_1 + \cdots + n_{k-1}$. By Lemma 2, r must be less than 2, and the only splittings occur in the complex projective space case. $\square$

4. Hard results: the case $\mathbf{F} = \mathbf{R}$.

LEMMA 3. *If the fibering*

$$\pi\colon \mathbf{R}M(1,1,m_1,\ldots,m_j) \to \mathbf{R}M(1,1,m)$$

with $j > 1$ has a section, then $(m_1, m_2,\ldots,m_j) = (1, m-1)$ and, in fact, $m = 5$.

PROOF. Suppose there is a section, so that γ_3 over $\mathbf{R}M(1,1,m)$ has a proper Whitney sum decomposition $\gamma_3 = \xi^p \oplus \eta^q$, $p + q = m$, $p \leq q$. Let $w_1(\gamma_1) = x$, $w_1(\gamma_2) = y$ be the first Stiefel-Whitney classes of the two line bundles. Then $H^*(\mathbf{R}M(1,1,m); Z_2)$ is the Z_2 polynomial ring on x and y modulo relations in dimensions $m+1$ and higher.

Claim. There are integers a, b, c with $w(\xi) = (1+x)^a(1+y)^b(1+x+y)^c$. To see this, one has maps

$$\begin{array}{ccc} \mathbf{R}P^\infty \times \mathbf{R}P^\infty & & \\ f\uparrow & & \\ \mathbf{R}M(1,1,m) & \xrightarrow{g} & BO_p \end{array}$$

where f classifies γ_1 and γ_2 and g classifies ξ^p. In cohomology one has

$$\begin{array}{ccc} H^*(\mathbf{R}P^\infty \times \mathbf{R}P^\infty; Z_2) & & \\ f^*\downarrow & & \\ H^*(\mathbf{R}M(1,1,m); Z_2) & \xleftarrow{g^*} & H^*(BO_p; Z_2) \end{array}$$

with f^* an isomorphism through dimension $m \geq 2p$. Now $H^*(BO_p; Z_2)$ is generated by $w_1, w_2,\ldots,w_p$ with relations given by the Wu formulae for $\mathrm{Sq}^i w_j$ $(1 \leq i \leq j)$, which occur in dimensions at most $2p$, and so one has a lifting

$$\begin{array}{ccc} H^*(\mathbf{R}P^\infty \times \mathbf{R}P^\infty; Z_2) & & \\ f^*\downarrow & & \nwarrow \phi \\ H^*(\mathbf{R}M(1,1,m); Z_2) & \xleftarrow{g^*} & H^*(BO_p; Z_2) \end{array}$$

with ϕ being a homomorphism of algebras over the Steenrod algebra. Now Patterson **[4]** shows that any such homomorphism

$$\tilde{\phi}\colon H^*(BO; Z_2) \to H^*(\mathbf{R}P^\infty \times \mathbf{R}P^\infty; Z_2)$$

has the form $\tilde{\phi}(w) = (1+x)^a(1+y)^b(1+x+y)^c$, which gives the claim.

Claim. $w(\xi)$ is symmetric.

To see this, let f_1, f_2: $\mathbf{R}M(1, m) \to \mathbf{R}M(1, 1, m)$ be the two inclusions, for which $f_1^*(\gamma_1) \cong \gamma^1, f_1^*(\gamma_2) \cong 1$ = trivial and $f_2^*(\gamma_1) = 1, f_2^*(\gamma_2) = \gamma^1$, and hence $f_i^*(\gamma_3) = \gamma^m \supset f_i^*(\xi^p)$. Let $z = w_1(\gamma^1)$, one has from [3] that $w(f_i^*(\xi^p)) = (1+z)^p$. Thus $(1+z)^a = (1+z)^p/(1+z)^c = (1+z)^b$ in $H^*(\mathbf{R}M(1, m); Z_2)$, so that the coefficient of x^t in $(1+x)^a$ is the same as that of y^t in $(1+y)^b$ for $t \leqslant m$. Thus interchanging x and y does not change $w_1(\xi), \ldots, w_p(\xi)$, and $w(\xi)$ is symmetric in x and y.

Thus $w(\xi^p)$ is symmetric in x and y, and hence comes from $H^*(\mathbf{R}M(2, m); Z_2)$. The arguments in [5] about splitting γ^m over $\mathbf{R}M(2, m)$ then apply to give $m = 5$ and $p = 1$ and also $w(\xi) = 1 + (x + y)$. Notice that Lemma 3 of [5] is purely algebraic by the note after the proof and that the geometric part of Lemma 4 about sectioning $O(n+2)/O(n-1) \to O(n+2)/O(n)$ occurs over the space of flags. □

LEMMA 4. *If the fibering*

$$\pi\colon \mathbf{R}M\Big(\underbrace{1,\ldots,1}_{r}, m_1, \ldots, m_j\Big) \to \mathbf{R}M\Big(\underbrace{1,\ldots,1}_{r}, m\Big)$$

with $j > 1$ *has a section, then* $r \leqslant 3$.

PROOF. By Lemma 3, $m = 5$, $(m_1, \ldots, m_j) = (1, 4)$, using Lemma 1 implicitly. Further, with Lemma 1, one may reduce the case $r \geqslant 4$ to $r = 4$. However, the argument for splitting γ^5 over $\mathbf{R}M(4, 5)$ in [5] shows that no decomposition can exist. □

PROPOSITION 3. *In the real case* ($\mathbf{F} = \mathbf{R}$), *the only genuine fiberings*

$$\pi\colon \mathbf{R}M\big(n_1^1, \ldots, n_{j_k}^k\big) \to \mathbf{R}M(n_1, \ldots, n_k)$$

$(j_k > 1)$ *which have sections are*:

$$\text{(a)}\quad \pi\colon \mathbf{R}M\big(1, n_1^k, \ldots, n_{j_k}^k\big) \to \mathbf{R}M(1, n_k),$$

which occur for n_k *odd and* $\sup(n_j^k) \geqslant (n_k + 1) - \rho(n_k + 1)$, *and*

$$\text{(b)}\quad \pi\colon \mathbf{R}M(n_1, \ldots, n_{k-1}, 1, 4) \to \mathbf{R}M(n_1, \ldots, n_{k-1}, 5)$$

with $n_1 + \cdots + n_{k-1} \leqslant 3$.

PROOF. By Lemmas 1 and 4, $n_1 + \cdots + n_{k-1} = r \leqslant 3$. The possible splittings for $r = 1$ were given in [3]. For $r \geqslant 2$, Lemma 3 gives $n_k = 5$, $(n_1^k, \ldots, n_{j_k}^k) = (1, 4)$. Pulling back the splitting of γ^5 over $\mathbf{R}M(3, 5)$, given in [5], gives all splittings of (b). It is clear that none of the bundles $\gamma_1, \ldots, \gamma_{k-1}$ over $\mathbf{R}M(n_1, \ldots, n_{k-1}, 5)$ with $n_1 + \cdots + n_{k-1} \leqslant 3$ can decompose. □

5. Real splittings for quaternionic Grassmannians. The purpose of this section is to complete the results of [5] by proving:

PROPOSITION 4. *Neither of the bundles* γ_k *and* γ_n *over* $G_k(\mathbf{H}^{n+k}) = \mathbf{H}M(k, n)$ *with* $2 \leqslant k \leqslant n$ *contains a proper real subbundle.*

Note. During the preparation of [5], I did not consider this case since it would easily follow if you could prove nonsplitting over $\mathbf{R}$ for γ_n over $\mathbf{H}P^n$. Unfortunately, that is difficult, if indeed it is true. Thus, it seems worthwhile to do the Grassmannian case.

PROOF. One has $H^*(G_k(\mathbf{H}^{n+k}); Z) = Z[p_1,\dots,p_k, \bar{p}_1,\dots,\bar{p}_n]/\{(p\bar{p}) = 1\}$ where $p = 1 + p_1 + \cdots + p_k = \mathcal{P}^s(\gamma_k)$ and $1 + \bar{p}_1 + \cdots + \bar{p}_n = \mathcal{P}^s(\gamma_n)$ are the symplectic Pontrjagin classes, so that

$$c(\gamma_k) = 1 - p_1 + p_2 + \cdots + (-1)^k p_k.$$

Since the Euler class $X(\gamma_k) = c_{2k}(\gamma_k) = (-1)^k p_k$ is indecomposable, γ_k admits no splitting.

If $\gamma_n = \xi^j \oplus \eta^{4n-j}$ over $G_k(\mathbf{H}^{n+k})$, then $\gamma_n = \xi^j \oplus \eta^{4n-j}$ over $G_2(\mathbf{H}^{n+2})$. Applying the mod 2 cohomology analysis [5] as for $G_2(\mathbf{R}^{n+2})$ but with degrees multiplied by 4, one has $n = 2^s - 3$ for some ($s \geqslant 3$) and $j = 4$. By properly choosing orientations

$$\begin{aligned} X(\xi^4) \cdot X(\eta^{4n-4}) &= X(\gamma_n) = c_{2n}(\gamma_n) = (-1)^n \bar{p}_n \\ &= \pm p_1^n + \text{terms divisible by } p_2 \end{aligned}$$

and so $X(\xi^4) = \pm p_1$ is not divisible by any integer.

Applying the mod 3 reduced power operation,

$$\mathcal{P}^1(p_1) = \mathcal{P}^1(x^2 + y^2) = 2x^4 + 2y^4 = 2\left((x^2 + y^2)^2 - 2x^2y^2\right) = 2p_1^2 - p_2$$

(where $p = (1 + x^2)(1 + y^2)$ via the splitting principle). Since this is not a multiple of p_1, no vector bundle can have Euler class equal to $\pm p_1$. $\square$

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