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A DUALITY PRINCIPLE

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ABSTRACT. With the aid of the Baire category theory we prove an extension of Erdös' well-known duality principle concerning sets of Lebesgue measure zero and sets of first category.

Assuming the continuum hypothesis, Sierpiński [9] proved the existence of a bijective function f: $\mathbf{R} \to \mathbf{R}$, such that A is of first category iff $f(A)$ has Lebesgue measure zero. By a modification of Sierpiński's proof, Erdös [2] showed that the function f in Sierpiński's result can be chosen such that $f = f^{-1}$. The importance of these results is that they allow one to state a well-known duality principle (see [6]). Using Morgan's abstract Baire category theory (cf. [3–5]), Cholewa [1] generalized Sierpiński's theorem, but an analogous generalization of Erdös' theorem is not known.

The aim of this paper is to give a short proof of a new extension of Erdös' result. In this note we assume, that X is a topological group, which is a complete, separable, metric space without isolated points; moreover we suppose that the reader is familiar with Morgan's theory.

THEOREM 1. *Let $c = \omega_1$. If $\mathcal{C}$ and $\mathfrak{D}$ are nonequivalent $\mathfrak{P}$-families [4] and $\mathfrak{S}$-families on X, satisfying c.c.c. (countable chain condition), then there is a bijective function f: $X \to X$ such that $f = f^{-1}$ and such that $A \in \mathcal{C}_{\mathrm{I}}$ iff $f(A) \in \mathfrak{D}_{\mathrm{I}}$.*

PROOF. The properties of X imply that $|X| = c$, since otherwise

$$X = \bigcup \{\{x\} : x \in X\}$$

would be of first category. We note that $\mathcal{C}_{\mathrm{I}}$ is a σ-ideal such that $X = \bigcup \mathcal{C}_{\mathrm{I}}$, since $\{x\} \in \mathcal{C}_{\mathrm{I}}$ for all $x \in X$ [4, Theorem 6]. Now let

$$\mathfrak{g} := \{A \in \mathcal{-C}_{\delta\sigma} \cap \mathcal{C}_{\mathrm{I}}: |A| > \aleph_0\},$$

where $-\mathcal{C}$ denotes the family of all sets being complements of $\mathcal{C}$-sets. By Theorem 3 in [5] and by Corollary 10 in [4] each $\mathcal{C}_{\mathrm{I}}$-set is contained in some $\mathfrak{g}$-set. Since $\mathcal{C}$ consists of perfect sets, we get $|\mathcal{C}| \leq c$ and thus $|\mathfrak{g}| \leq c$. Moreover, the complement of a $\mathcal{C}_{\mathrm{I}}$-set lies in $\mathfrak{B}(\mathcal{C}) \cap \mathcal{C}_{\mathrm{II}}$ and by Corollary 15 in [4], it contains a nonempty $\mathcal{C}$-singular perfect set, that is, it contains a $\mathcal{C}_{\mathrm{I}}$-set of power c (Lemma 3 in [4] shows that, without loss of generality, we can assume, that $X \in \mathcal{C} \subset \mathcal{C}_{\mathrm{II}}$).

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Thus $\mathcal{C}_{\mathrm{I}}$ fulfills the properties (a)–(d) of Theorem 19.5 in [**6**]. It is clear, that the same fact is true for $\mathfrak{D}_{\mathrm{I}}$. Since $\mathcal{C}$ and $\mathfrak{D}$ are nonequivalent, it follows from Theorem 2 in [**8**], that X is the disjoint union of a $\mathcal{C}_{\mathrm{I}}$-set and a $\mathfrak{D}_{\mathrm{I}}$-set. Thus, by Theorem 19.6 in [**6**], the proof is finished.

The classical result of Erdös can be obtained from Theorem 1 by setting $\mathcal{C} := \{\{x \in \mathbf{R}\colon |x-y| \leqslant 1/n\}\colon y \in \mathbf{R},\ n \in \mathbf{N}\}$ and $\mathfrak{D} := \{A \subset \mathbf{R}\colon A$ is closed, $\forall x \in A\ \forall U \in \tau x\colon \mu(U \cap A) > 0\}$ (τx consists of all open subsets of $\mathbf{R}$ containing x and μ denotes the Borel measure in $\mathbf{R}$). It follows, that $\mathcal{C}$ and $\mathfrak{D}$ are nonequivalent $\mathfrak{P}$- and $\mathfrak{S}$-families (see [**4** and **8**]). Since the members of $\mathcal{C}$ and $\mathfrak{D}$ have positive measure, $\mathcal{C}$ and $\mathfrak{D}$ also satisfy c.c.c. [**7**, p. 123].

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