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A NEW PROOF OF A THEOREM OF SOLOMON-TITS[1]

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ABSTRACT. Let Δ be the combinatorial building of a finite group of Lie type G. A new proof is given of the theorem of Solomon-Tits on the G-module structure of the rational homology $H_*(\Delta)$ of Δ.

Let G be a finite group with a Tits system (B, N, W, R) (Bourbaki [**1**, Chapter 4]), with $|R| = n \geq 2$. The *combinatorial building* Δ of G is the simplicial complex whose simplices are the proper parabolic subgroups of G, ordered by the opposite of the inclusion relation. The group G acts on Δ by conjugation, and on the rational homology $H_*(\Delta)$ of Δ. In this note we shall prove

THEOREM (SOLOMON-TITS). *The rational homology of Δ is zero except in dimensions* 0 *and* $n-1$, *and in these dimensions affords the trivial representation* 1_G, *and a nontrivial absolutely irreducible representation* St_G (*the Steinberg representation*), *respectively.*

The previous proofs of this theorem (Solomon [**7**], Garland [**6**]) show that Δ has the homotopy type of a bouquet of spheres. We give a new proof based on the computation of the QG-endomorphism algebra of $H_*(\Delta)$.

We begin with some topological preliminaries. Let X be a finite simplicial complex whose vertices form a partially ordered set (abbreviation: poset), and whose simplices are the chains in that poset. Let G be a finite group acting on the vertices of X, preserving the order relation, and hence the simplicial structure. We assume that G acts regularly on the vertices, so that $x \leq y$, $gx \leq y$ imply $x = gx$, for all vertices x, y and $g \in G$ (cf. Bredon [**2**], Curtis-Lehrer [**4**]). The Cartesian product $X \times X$ is the simplicial complex arising from the poset whose elements are the Cartesian product with itself of the vertex set of X, with order relation $(x, y) \leq (x', y')$ if $x \leq x'$, $y \leq y'$, and diagonal G-action. If X is regular, then so is $X \times X$, and the orbit simplicial complex $G\backslash(X \times X)$ is defined. We have

PROPOSITION. *Let X be a simplicial complex, with a regular G-action. There is a graded isomorphism of vector spaces over Q,*

$$\mathrm{End}_{QG}(H_*(X)) \simeq H_*(G\backslash(X \times X)),$$

with

$$H_r(G\backslash(X \times X)) \simeq \bigoplus_{j=0}^{r} \mathrm{Hom}_{QG}(H_j(X), H_{r-j}(X)), \quad r \geq 0.$$

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For a proof and further discussion, see Curtis [**3**] and Curtis-Lehrer [**4**]. The application to $H_*(\Delta)$ is based on a comparison with the *Coxeter complex* Γ of the Weyl group W of G, which is the simplicial complex whose simplices are all cosets $\{wW_J : w \in W\}$ of proper parabolic subgroups W_J of W, ordered by the opposite of inclusion, with W-action given by left translation.

To apply the proposition directly, we actually work later with the barycentric subdivisions Δ' and Γ' of Δ and Γ respectively. These are the simplicial complexes arising from the *posets* Δ and Γ. It is well known and standard (cf. [**4**, §2]) that for any simplicial complex K, if K' is its barycentric subdivision (i.e. the complex arising from the poset of simplices of K) then $H_*(K)$ and $H_*(K')$ are canonically isomorphic, and the isomorphism preserves any group action which may be present.

LEMMA 1. *The rational homology $H_*(\Gamma)$ is zero except in dimensions* 0 *and* $n-1$, *and in these dimensions affords the trivial representation* 1_W *and the sign representation* $\epsilon\colon w \to (-1)^{\ell(w)}$, $w \in W$ (*where $\ell(w)$ is the length of w*), *respectively.*

The proof follows directly from the fact that the underlying topological space $|\Gamma|$ of Γ is a sphere (Bourbaki [**1**, Chapter 5]).

We next observe that Γ and Δ are posets with regular group actions by W and G, respectively, so that the orbit posets $W\backslash(\Gamma \times \Gamma)$ and $G\backslash(\Delta \times \Delta)$ are defined. The key result is

LEMMA 2. *There exists an isomorphism of posets* $W\backslash(\Gamma \times \Gamma) \simeq G\backslash(\Delta \times \Delta)$.

For subsets $J_1, J_2 \subset R$, let $X(J_1, J_2)$ be the set of distinguished (shortest) double coset representatives for $W_{J_1}\backslash W/W_{J_2}$ (Bourbaki [**1**, Chapter 4]). We shall prove that both posets in the statements of the lemma are isomorphic to the poset Ω consisting of all triples $\omega = (J_1, J_2, w)$, $J_1, J_2 \subset R$, $w \in X(J_1, J_2)$, with

$$(J_1, J_2, w) \leq (J_1', J_2', w') \quad \text{if } J_1 \supseteq J_1', J_2 \supseteq J_2', W_{J_1}wW_{J_2} \supseteq W_{J_1'}w'W_{J_2'}.$$

Let $\mathcal{O}$ be a G-orbit in $\Delta \times \Delta$; then $\mathcal{O}$ contains $(P_{J_1}, {}^gP_{J_2})$ for some proper standard parabolic subgroups P_{J_1} and P_{J_2}. We have $g \in P_{J_1}wP_{J_2}$ for some $w \in X(J_1, J_2)$, so $\mathcal{O}$ contains $(P_{J_1}, {}^wP_{J_2})$, and corresponds to $\omega = (J_1, J_2, w) \in \Omega$. For two orbits $\mathcal{O}, \mathcal{O}'$ containing representatives $(P_{J_1}, {}^wP_{J_2})$ and $(P_{J_1'}, {}^{w'}P_{J_2'})$ respectively, with $w \in X(J_1, J_2)$, $w' \in X(J_1', J_2')$, we have (recalling that the order relation in Δ is the opposite of inclusion), $\mathcal{O} \leq \mathcal{O}' \Leftrightarrow P_{J_1} \supseteq {}^gP_{J_1'}$, ${}^wP_{J_2} \supseteq {}^{gw'}P_{J_2'}$, for some $g \in G \Leftrightarrow g \in P_{J_1}$, $w^{-1}gw' \in P_{J_2}$ (by Bourbaki, loc. cit.) $\Leftrightarrow J_1 \supseteq J_1'$, $J_2 \supseteq J_2'$ and $P_{J_1}wP_{J_2} \supseteq P_{J_1'}w'P_{J_2'}$. The latter condition is equivalent to $W_{J_1}wW_{J_2} \supseteq W_{J_1'}w'W_{J_2'}$, again by Bourbaki [**1**], and we have proved that $G\backslash(\Delta \times \Delta) \simeq \Omega$. The proof that $W\backslash(\Gamma \times \Gamma) \simeq \Omega$ is easier, and is left as an exercise.

PROOF OF THE THEOREM. As remarked after the proposition above, we have equivariant isomorphisms: $H_*(\Delta) \to H_*(\Delta')$ and $H_*(\Gamma) \to H_*(\Gamma')$ of graded modules, where Δ' is the barycentric subdivision of Δ and similarly for Γ'. Moreover, from the proposition we have a graded isomorphism: $\mathrm{End}_{QG}\, H_*(\Delta') \to H_*(G\backslash(\Delta' \times \Delta'))$ and a similar result for $\mathrm{End}_{QW}\, H_*(\Gamma')$. But $G\backslash(\Delta' \times \Delta')$ is the simplicial complex arising from the poset $G\backslash(\Delta \times \Delta)$. Thus from Lemma 2 and a similar remark about $W\backslash(\Gamma' \times \Gamma')$, we obtain that

$$H_*(G\backslash(\Delta' \times \Delta')) \simeq H_*(W\backslash(\Gamma' \times \Gamma')).$$

Composing these isomorphisms, we obtain a graded isomorphism

$$\mathrm{End}_{QG}(H_*(\Delta)) \to \mathrm{End}_{QW}(H_*(\Gamma)).$$

By Lemma 1, we have

$$\mathrm{End}_{QW} H_0(\Gamma) \simeq \mathrm{End}_{QW}(H_{n-1}(\Gamma)) \simeq Q \cdot 1,$$

and

$$\mathrm{Hom}_{QW}(H_i(\Gamma), H_j(\Gamma)) = 0 \quad \text{if } i + j \neq 0,\ 2(n-1).$$

By the above isomorphism, the same results hold for $H_*(\Delta)$. The first implies that the representations of G on $H_0(\Delta)$ and $H_{n-1}(\Delta)$ are absolutely irreducible, and the second that these are inequivalent, and that $H_i(\Delta) = 0$ for $i \neq 0$, $n-1$. Finally, it is easily checked that $H_0(\Delta)$ affords 1_G, and the theorem is proved.

Using the Hopf trace formula, it follows readily that the character of St_G is given by $\sum_{J \subseteq R} (-1)^{|J|} 1_{P_J}^G$, and hence that its degree is $|B \colon B \cap {}^{w_0}B|$, where w_0 is the element of maximal length in W. Using a related homological description of St_G, a character formula for St_G can be obtained in [**5**].

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